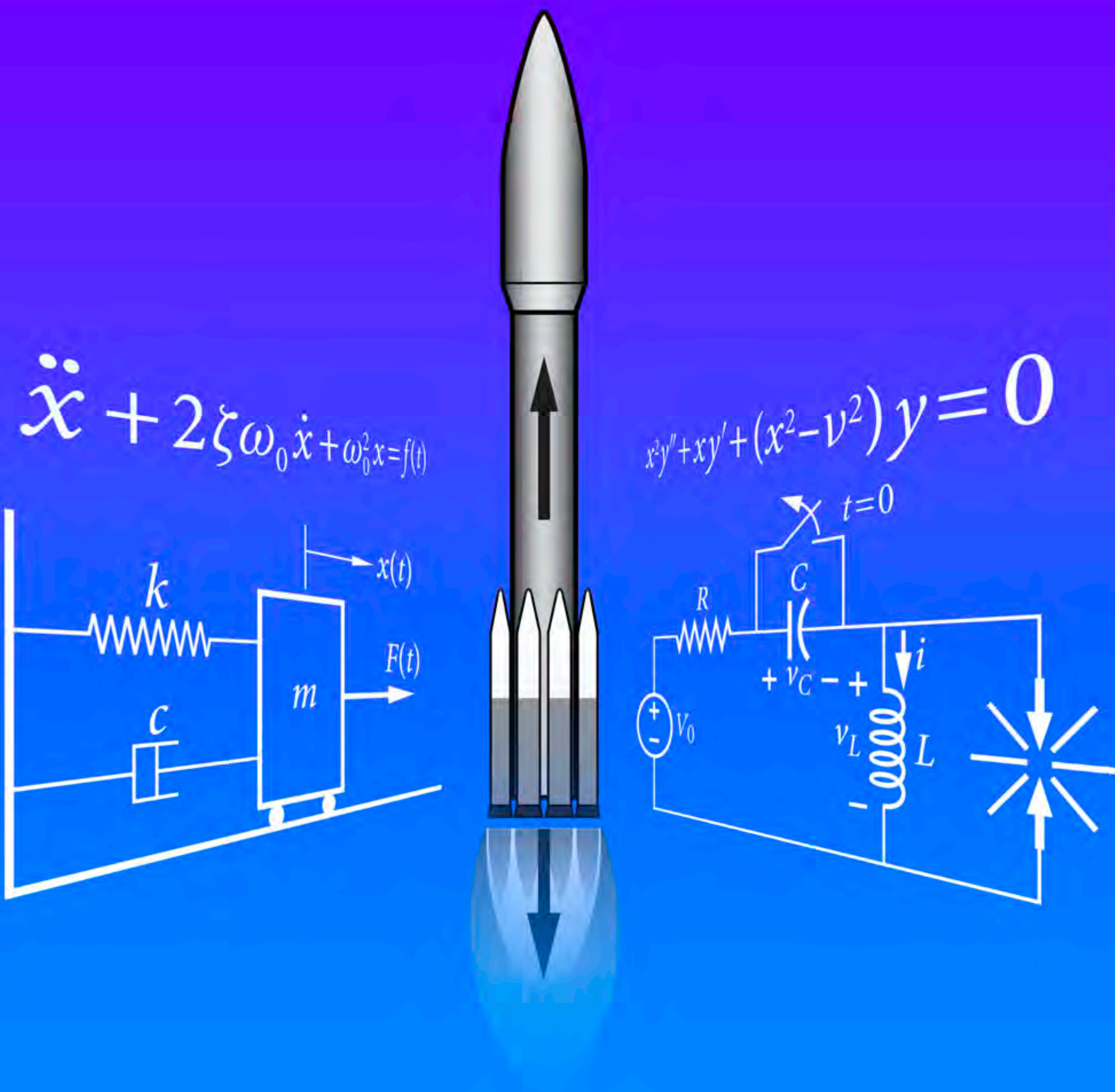
Differential Equations for Engineers



Wei-Chau Xie

DIFFERENTIAL EQUATIONS FOR ENGINEERS

This book presents a systematic and comprehensive introduction to ordinary differential equations for engineering students and practitioners. Mathematical concepts and various techniques are presented in a clear, logical, and concise manner. Various visual features are used to highlight focus areas. Complete illustrative diagrams are used to facilitate mathematical modeling of application problems. Readers are motivated by a focus on the relevance of differential equations through their applications in various engineering disciplines. Studies of various types of differential equations are determined by engineering applications. Theory and techniques for solving differential equations are then applied to solve practical engineering problems. Detailed step-by-step analysis is presented to model the engineering problems using differential equations from physical principles and to solve the differential equations using the easiest possible method. Such a detailed, step-by-step approach, especially when applied to practical engineering problems, helps the readers to develop problem-solving skills.

This book is suitable for use not only as a textbook on ordinary differential equations for undergraduate students in an engineering program but also as a guide to self-study. It can also be used as a reference after students have completed learning the subject.

Wei-Chau Xie is a Professor in the Department of Civil and Environmental Engineering and the Department of Applied Mathematics at the University of Waterloo. He is the author of *Dynamic Stability of Structures* and has published numerous journal articles on dynamic stability, structural dynamics and random vibration, nonlinear dynamics and stochastic mechanics, reliability and safety analysis of engineering systems, and seismic analysis and design of engineering structures. He has been teaching differential equations to engineering students for almost twenty years. He received the Teaching Excellence Award in 2001 in recognition of his exemplary record of outstanding teaching, concern for students, and commitment to the development and enrichment of engineering education at Waterloo. He is the recipient of the Distinguished Teacher Award in 2007, which is the highest formal recognition given by the University of Waterloo for a superior record of continued excellence in teaching.

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Preface

Background

Differential equations have wide applications in various engineering and science disciplines. In general, modeling of the variation of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, current, voltage, or concentration of a pollutant, with the change of time or location, or both would result in differential equations. Similarly, studying the variation of some physical quantities on other physical quantities would also lead to differential equations. In fact, many engineering subjects, such as mechanical vibration or structural dynamics, heat transfer, or theory of electric circuits, are founded on the theory of differential equations. It is practically important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behavior of the systems concerned can be studied.

I have been teaching differential equations to engineering students for the past two decades. Most, if not all, of the textbooks are written by mathematicians with little engineering background. Based on my experience and feedback from students, the following lists some of the gaps frequently seen in current textbooks:

❧ A major focus is put on explaining mathematical concepts

For engineers, the purpose of learning the theory of differential equations is to be able to solve practical problems where differential equations are used. For engineering students, it is more important to know the applications and techniques for solving application problems than to delve into the nuances of mathematical concepts and theorems. Knowing the appropriate applications can motivate them to study the mathematical concepts and techniques. However, it is much more challenging to model an application problem using physical principles and then solve the resulting differential equations than it is to merely carry out mathematical exercises.

❧ Insufficient emphasis is placed on the step-by-step problem solving techniques

Engineering students do not usually have the same mathematical background and interest as students who major in mathematics. Mathematicians are more interested if: (1) there are solutions to a differential equation or a system of differential equations; (2) the solutions are unique under a certain set of conditions; and (3) the differential equations can be solved. On the other hand, engineers are more interested in mathematical modeling of a practical problem and actually solving the equations to find the solutions using the easiest possible method. Hence, a detailed step-by-step approach, especially applied to practical engineering problems, helps students to develop problem solving skills.

- Presentations are usually formula-driven with little variation in visual design

  It is very difficult to attract students to read boring formulas without variation of presentation. Readers often miss the points of importance.

Objectives

This book addresses the needs of engineering students and aims to achieve the following objectives:

- To motivate students on the relevance of differential equations in engineering through their applications in various engineering disciplines. Studies of various types of differential equations are motivated by engineering applications; theory and techniques for solving differential equations are then applied to solve practical engineering problems.
- To have a balance between theory and applications. This book could be used as a reference after students have completed learning the subject. As a reference, it has to be reasonably comprehensive and complete. Detailed step-by-step analysis is presented to model the engineering problems using differential equations and to solve the differential equations.
- To present the mathematical concepts and various techniques in a clear, logical and concise manner. Various visual features, such as side-notes (preceded by the ✍ symbol), different fonts and shades, are used to highlight focus areas. Complete illustrative diagrams are used to facilitate mathematical modeling of application problems. This book is not only suitable as a textbook for classroom use but also is easy for self-study. As a textbook, it has to be easy to understand. For self-study, the presentation is detailed with all necessary steps and useful formulas given as side-notes.

Scope

This book is primarily for engineering students and practitioners as the main audience. It is suitable as a textbook on ordinary differential equations for undergraduate students in an engineering program. Such a course is usually offered in the second year after students have taken calculus and linear algebra in the first year. Although it is assumed that students have a working knowledge of calculus and linear algebra, some important concepts and results are reviewed when they are first used so as to refresh their memory.

Chapter 1 first presents some motivating examples, which will be studied in detail later in the book, to illustrate how differential equations arise in engineering applications. Some basic general concepts of differential equations are then introduced.

In Chapter 2, various techniques for solving first-order and simple higher-order ordinary differential equations are presented. These methods are then applied in Chapter 3 to study various application problems involving first-order and simple higher-order differential equations.

Chapter 4 studies linear ordinary differential equations. Complementary solutions are obtained through the characteristic equations and characteristic numbers. Particular solutions are obtained using the method of undetermined coefficients, the operator method, and the method of variation of parameters. Applications involving linear ordinary differential equations are presented in Chapter 5.

Solutions of linear ordinary differential equations using the Laplace transform are studied in Chapter 6, emphasizing functions involving Heaviside step function and Dirac delta function.

Chapter 7 studies solutions of systems of linear ordinary differential equations. The method of operator, the method of Laplace transform, and the matrix method are introduced. Applications involving systems of linear ordinary differential equations are considered in Chapter 8.

In Chapter 9, solutions of ordinary differential equations in series about an ordinary point and a regular singular point are presented. Applications of Bessel's equation in engineering are considered.

Some classical methods, including forward and backward Euler method, improved Euler method, and Runge-Kutta methods, are presented in Chapter 10 for numerical solutions of ordinary differential equations.

In Chapter 11, the method of separation of variables is applied to solve partial differential equations. When the method is applicable, it converts a partial differential equation into a set of ordinary differential equations. Flexural vibration of beams and heat conduction are studied as examples of application.

Solutions of ordinary differential equations using *Maple* are presented in Chapter 12. Symbolic computation software, such as *Maple*, is very efficient in solving problems involving ordinary differential equations. However, it cannot replace learning and thinking, especially mathematical modeling. It is important to develop analytical skills and proficiency through "hand" calculations, as has been done in previous chapters. This will also help the development of insight into the problems and appreciation of the solution process. For this reason, solutions of ordinary differential equations using *Maple* is presented in the last chapter of the book instead of a scattering throughout the book.

The book covers a wide range of materials on ordinary differential equations and their engineering applications. There are more than enough materials for a one-term (semester) undergraduate course. Instructors can select the materials according to the curriculum. Drafts of this book were used as the textbook in a one-term undergraduate course at the University of Waterloo.

Acknowledgments

First and foremost, my sincere appreciation goes to my students. It is the students who give me a stage where I can cultivate my talent and passion for teaching. It is for the students that this book is written, as my small contribution to their success in academic and professional careers. My undergraduate students who have used the draft of this book as a textbook have made many encouraging comments and constructive suggestions.

I am very grateful to many people who have reviewed and commented on the book, including Professor Hong-Jian Lai of West Virginia University, Professors S.T. Ariaratnam, Xin-Zhi Liu, Stanislav Potapenko, and Edward Vrscay of the University of Waterloo.

My graduate students Mohamad Alwan, Qinghua Huang, Jun Liu, Shunhao Ni, and Richard Wiebe have carefully read the book and made many helpful and critical suggestions.

My sincere appreciation goes to Mr. Peter Gordon, Senior Editor, Engineering, Cambridge University Press, for his encouragement, trust, and hard work to publish this book.

Special thanks are due to Mr. John Bennett, my mentor, teacher, and friend, for his advice and guidance. He has also painstakingly proofread and copyedited this book.

Without the unfailing love and support of my mother, who has always believed in me, this work would not have been possible. In addition, the care, love, patience, and understanding of my wife Cong-Rong and lovely daughters Victoria and Tiffany have been of inestimable encouragement and help. I love them very much and appreciate all that they have contributed to my work.

I appreciate hearing your comments through email (xie@uwaterloo.ca) or regular correspondence.

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CHAPTER 1

Introduction

1.1 Motivating Examples

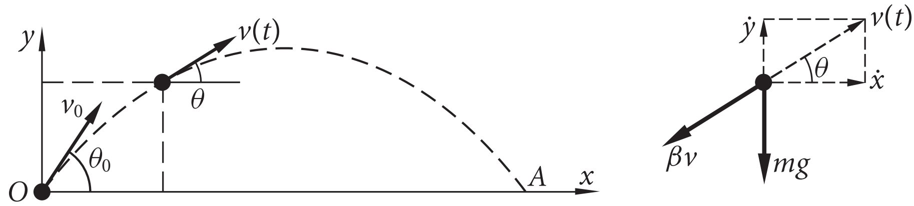
Differential equations have wide applications in various engineering and science disciplines. In general, modeling variations of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, or concentration of a pollutant, with the change of time t or location, such as the coordinates (x, y, z), or both would require differential equations. Similarly, studying the variation of a physical quantity on other physical quantities would lead to differential equations. For example, the change of strain on stress for some viscoelastic materials follows a differential equation.

It is important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behavior of the systems concerned can be studied.

In this section, a few examples are presented to illustrate how practical problems are modeled mathematically and how differential equations arise in them.

Motivating Example 1

First consider the projectile of a mass m launched with initial velocity v_0 at angle θ_0 at time $t = 0$, as shown.



The atmosphere exerts a resistance force on the mass, which is proportional to the instantaneous velocity of the mass, i.e., $R=\beta v$, where β is a constant, and is opposite to the direction of the velocity of the mass. Set up the Cartesian coordinate system as shown by placing the origin at the point from where the mass m is launched.

At time t, the mass is at location $\big(x(t), y(t)\big)$. The instantaneous velocity of the mass in the x- and y-directions are $\dot{x}(t)$ and $\dot{y}(t)$, respectively. Hence the velocity of the mass is $v(t)=\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}$ at the angle $\theta(t)=\tan^{-1}\big[\dot{y}(t)/\dot{x}(t)\big]$.

The mass is subjected to two forces: the vertical downward gravity mg and the resistance force $R(t)=\beta v(t)$.

The equations of motion of the mass can be established using Newton's Second Law: $F=\sum ma$. The x-component of the resistance force is $-R(t)\cos\theta(t)$. In the y-direction, the component of the resistance force is $-R(t)\sin\theta(t)$. Hence, applying Newton's Second Law yields

$$x\text{-direction:}\quad ma_x=\sum F_x \quad\Longrightarrow\quad m\ddot{x}(t) = -R(t)\cos\theta(t),$$

$$y\text{-direction:}\quad ma_y=\sum F_y \quad\Longrightarrow\quad m\ddot{y}(t) = -mg - R(t)\sin\theta(t).$$

Since

$$\theta(t) = \tan^{-1}\frac{\dot{y}(t)}{\dot{x}(t)} \quad\Longrightarrow\quad \cos\theta = \frac{\dot{x}(t)}{\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}}, \qquad \sin\theta = \frac{\dot{y}(t)}{\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}},$$

the equations of motion become

$$m\ddot{x}(t) = -\beta v(t)\cdot\frac{\dot{x}(t)}{\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}} \quad\Longrightarrow\quad m\ddot{x}(t)+\beta\dot{x}(t)=0,$$

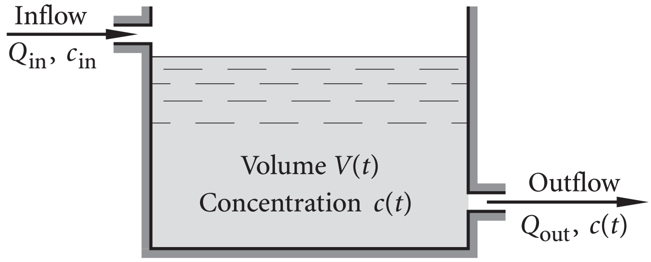
$$m\ddot{y}(t) = -mg-\beta v(t)\cdot\frac{\dot{y}(t)}{\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}} \quad\Longrightarrow\quad m\ddot{y}(t)+\beta\dot{y}(t)=-mg,$$

in which the initial conditions are at time $t=0$: $x(0)=0$, $y(0)=0$, $\dot{x}(0)=v_0\cos\theta_0$, $\dot{y}(0)=v_0\sin\theta_0$. The equations of motion are two equations involving the first- and second-order derivatives $\dot{x}(t)$, $\dot{y}(t)$, $\ddot{x}(t)$, and $\ddot{y}(t)$. These equations are called, as will be defined later, a system of two second-order ordinary differential equations.

Because of the complexity of the problems, in the following examples, the problems are described and the governing equations are presented without detailed derivation. These problems will be investigated in details in later chapters when applications of various types of differential equations are studied.

Motivating Example 2

A tank contains a liquid of volume $V(t)$, which is polluted with a pollutant concentration in *percentage* of $c(t)$ at time t. To reduce the pollutant concentration, an inflow of rate Q_{in} is injected to the tank. Unfortunately, the inflow is also polluted but to a lesser degree with a pollutant concentration c_{in}. It is assumed that the inflow is perfectly mixed with the liquid in the tank instantaneously. An outflow of rate Q_{out} is removed from the tank as shown. Suppose that, at time $t=0$, the volume of the liquid is V_0 with a pollutant concentration of c_0.

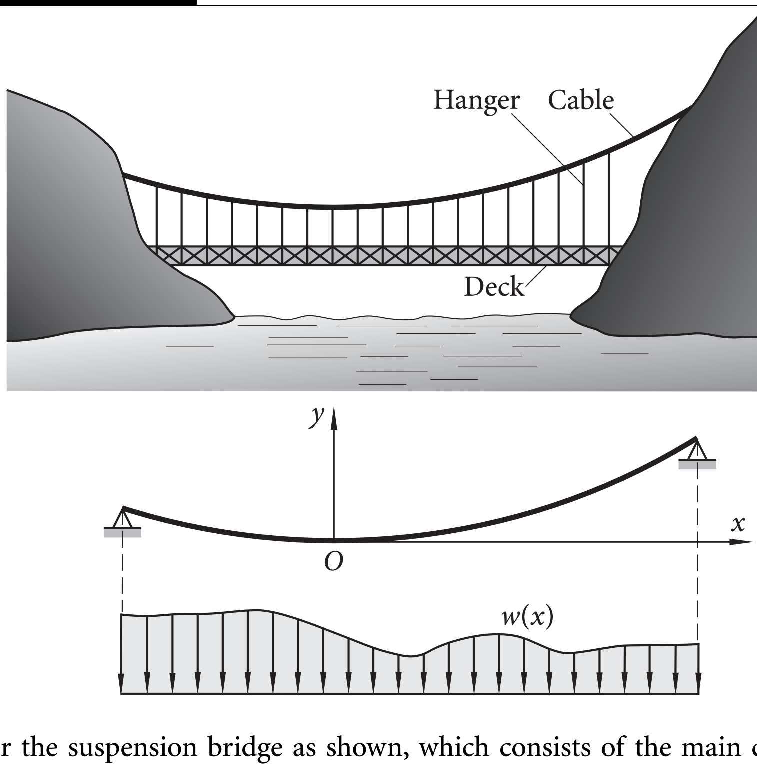


The equation governing the pollutant concentration $c(t)$ is given by

$$\left[V_0 + (Q_{\text{in}} - Q_{\text{out}})\,t\right]\frac{\mathrm{d}c(t)}{\mathrm{d}t} + Q_{\text{in}}\,c(t) = Q_{\text{in}}\,c_{\text{in}},$$

with initial condition $c(0)=c_0$. This is a first-order ordinary differential equation.

Motivating Example 3



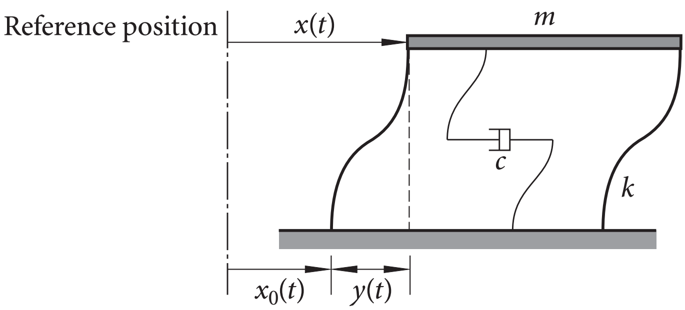
Consider the suspension bridge as shown, which consists of the main cable, the hangers, and the deck. The self-weight of the deck and the loads applied on the deck are transferred to the cable through the hangers.

Set up the Cartesian coordinate system by placing the origin O at the lowest point of the cable. The cable can be modeled as subjected to a distributed load $w(x)$. The equation governing the shape of the cable is given by

$$\frac{d^2y}{dx^2} = \frac{w(x)}{H},$$

where H is the tension in the cable at the lowest point O. This is a second-order ordinary differential equation.

Motivating Example 4



Consider the vibration of a single-story shear building under the excitation of earthquake. The shear building consists of a rigid girder of mass m supported by columns of combined stiffness k. The vibration of the girder can be described by the horizontal displacement $x(t)$. The earthquake is modeled by the displacement of the ground $x_0(t)$ as shown. When the girder vibrates, there is a damping force due to the internal friction between various components of the building, given by $c\left[\dot{x}(t) - \dot{x}_0(t)\right]$, where c is the damping coefficient.

The relative displacement $y(t) = x(t) - x_0(t)$ between the girder and the ground is governed by the equation

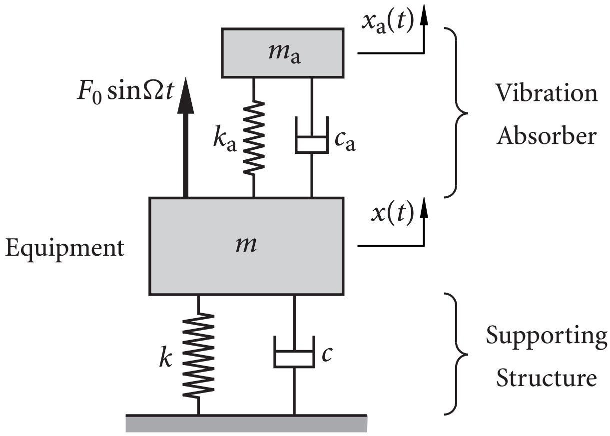
$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = -m\ddot{x}_0(t),$$

which is a second-order linear ordinary differential equation.

Motivating Example 5

In many engineering applications, an equipment of mass m is usually mounted on a supporting structure that can be modeled as a spring of stiffness k and a damper of damping coefficient c as shown in the following figure. Due to unbalanced mass in rotating components or other excitation mechanisms, the equipment is subjected to a harmonic force $F_0 \sin \Omega t$. The vibration of the mass is described by the vertical displacement $x(t)$. When the excitation frequency Ω is close to $\omega_0 = \sqrt{k/m}$, which is the natural circular frequency of the equipment and its support, vibration of large amplitudes occurs.

In order to reduce the vibration of the equipment, a vibration absorber is mounted on the equipment. The vibration absorber can be modeled as a mass m_a, a spring of stiffness k_a, and a damper of damping coefficient c_a. The vibration of the absorber is described by the vertical displacement $x_a(t)$.



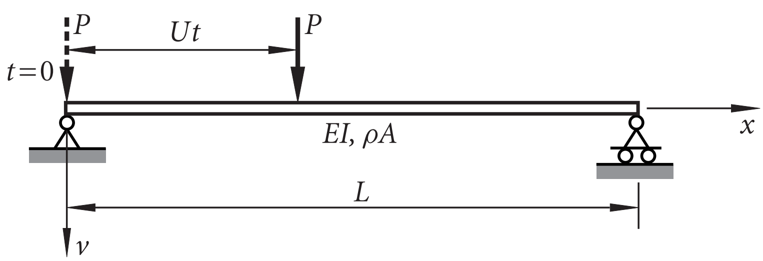
The equations of motion governing the vibration of the equipment and the absorber are given by

$$m\ddot{x} + (c+c_a)\dot{x} + (k+k_a)x - c_a\dot{x}_a - k_a x_a = F_0 \sin \Omega t,$$

$$m_a\ddot{x}_a + c_a\dot{x}_a + k_a x_a - c_a\dot{x} - k_a x = 0,$$

which comprises a system of two coupled second-order linear ordinary differential equations.

Motivating Example 6



A bridge may be modeled as a simply supported beam of length L, mass density per unit length ρA, and flexural rigidity EI as shown. A vehicle of weight P crosses the bridge at a constant speed U. Suppose at time $t=0$, the vehicle is at the left end of the bridge and the bridge is at rest. The deflection of the bridge is $v(x,t)$, which is a function of both location x and time t. The equation governing $v(x,t)$ is the partial differential equation

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + EI \frac{\partial^4 v(x,t)}{\partial x^4} = P\delta(x-Ut),$$

Example 2.19

Solve $\quad y\,(\cos^3 x + y \sin x)\,\mathrm{d}x + \cos x\,(\sin x \cos x + 2y)\,\mathrm{d}y = 0.$

The differential equation is of the standard form $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$, where

$$M(x, y) = y \cos^3 x + y^2 \sin x, \quad N(x, y) = \sin x \cos^2 x + 2y \cos x.$$

Test for exactness:

$$\frac{\partial M}{\partial y} = \cos^3 x + 2y \sin x, \qquad \frac{\partial N}{\partial x} = \cos^3 x - 2\sin^2 x \cos x - 2y \sin x,$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \Longrightarrow \quad \text{The differential equation is } \textit{not} \text{ exact.}$$

Since

$$\frac{1}{N}\Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big) = \frac{(\cos^3 x + 2y \sin x) - (\cos^3 x - 2\sin^2 x \cos x - 2y \sin x)}{\sin x \cos^2 x + 2y \cos x}$$

$$= \frac{2 \sin x\,(2y + \sin x \cos x)}{\cos x\,(2y + \sin x \cos x)} = \frac{2 \sin x}{\cos x}, \quad \text{✍ A function of } x \text{ only}$$

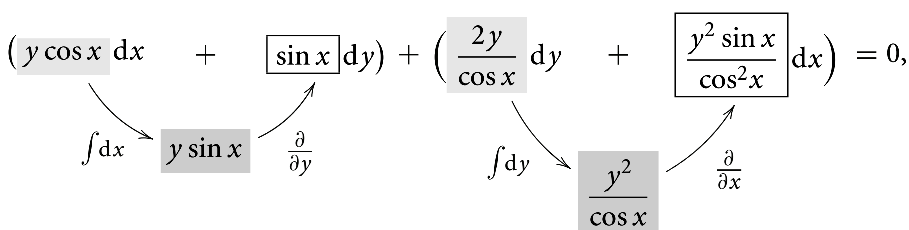
$$\therefore \quad \mu(x) = \exp\Big[\int \frac{1}{N}\Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big)\mathrm{d}x\Big] = \exp\Big[\int \frac{2 \sin x}{\cos x}\mathrm{d}x\Big]$$

$$= \exp\Big[-2\int \frac{1}{\cos x}\mathrm{d}(\cos x)\Big] = \exp\big[-2 \ln|\cos x|\big] = \frac{1}{\cos^2 x}.$$

Multiplying the differential equation by the integrating factor $\mu(x) = \dfrac{1}{\cos^2 x}$ yields

$$\Big(y \cos x + \frac{y^2 \sin x}{\cos^2 x}\Big)\mathrm{d}x + \Big(\sin x + \frac{2y}{\cos x}\Big)\mathrm{d}y = 0.$$

The general solution is determined using the method of grouping terms



which gives

$$y \sin x + \frac{y^2}{\cos x} = C. \quad \text{✍ General solution}$$

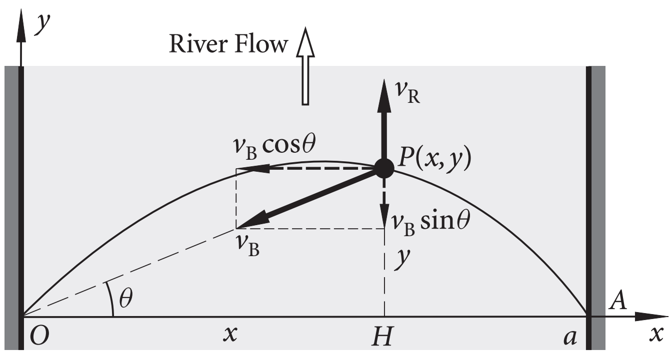
Procedure for Solving an Application Problem

1. Establish the governing differential equations based on physical principles and geometrical properties underlying the problem.
2. Identify the type of these differential equations and then solve them.
3. Determine the arbitrary constants in the general solutions using the initial or boundary conditions.

3.6 Various Application Problems

Example 3.10 — Ferry Boat

A ferry boat is crossing a river of width a from point A to point O as shown in the following figure. The boat is always aiming toward the destination O. The speed of the river flow is constant v_{R} and the speed of the boat is constant v_{B}. Determine the equation of the path traced by the boat.



Suppose that, at time t, the boat is at point P with coordinates (x, y). The velocity of the boat has two components: the velocity of the boat v_{B} relative to the river flow (as if the river is not flowing), which is pointing toward the origin O or along line PO, and the velocity of the river v_{R} in the y direction.

Decompose the velocity components v_B and v_R in the x- and y-directions

$$v_x = -v_B \cos\theta, \qquad v_y = v_R - v_B \sin\theta.$$

From $\triangle OHP$, it is easy to see

$$\cos\theta = \frac{OH}{OP} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\theta = \frac{PH}{OP} = \frac{y}{\sqrt{x^2+y^2}}.$$

Hence, the equations of motion are given by

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = -v_{\mathrm{B}} \frac{x}{\sqrt{x^2+y^2}}, \qquad v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = v_{\mathrm{R}} - v_{\mathrm{B}} \frac{y}{\sqrt{x^2+y^2}}.$$

Example 3.11 — Bar with Variable Cross-Section

A bar with circular cross-sections is supported at the top end and is subjected to a load of P as shown in Figure 3.14(a). The length of the bar is L. The weight density of the materials is ρ per unit volume. It is required that the stress at every point is constant σ_a. Determine the equation for the cross-section of the bar.

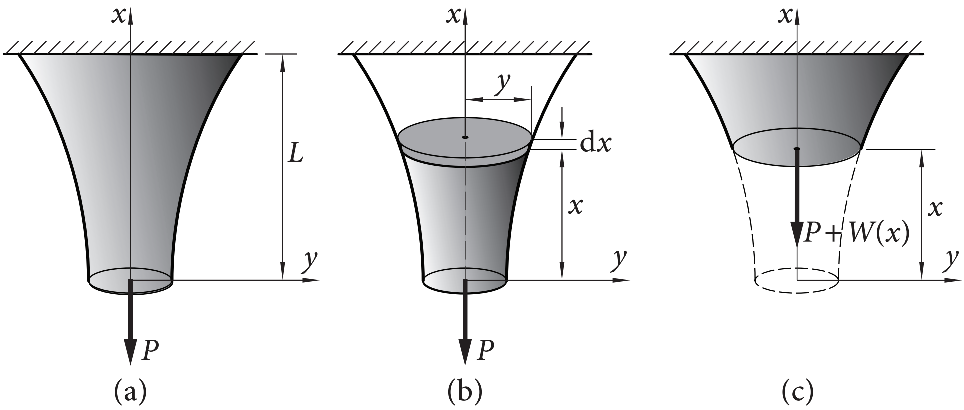


Figure 3.14 A bar under axial load.

Consider a cross-section at level x as shown in Figure 3.14(b). The corresponding radius is y. The volume of a circular disk of thickness $\mathrm{d}x$ is $\mathrm{d}V = \pi y^2 \mathrm{d}x$. The volume of the segment of bar between 0 and x is

$$V(x) = \int_0^x \pi y^2 \mathrm{d}x,$$

and the weight of this segment is

$$W(x) = \rho V(x) = \rho \int_0^x \pi y^2 \mathrm{d}x.$$

The load applied on cross-section at level x is equal to the sum of the externally applied load P and the weight of the segment between 0 and x, i.e.,

$$F(x) = W(x) + P = \rho \int_0^x \pi y^2 \mathrm{d}x + P.$$

The normal stress is

$$\sigma(x) = \frac{F(x)}{A(x)} = \frac{1}{\pi y^2}\Big(\rho \int_0^x \pi y^2 \mathrm{d}x + P\Big) = \sigma_a \implies \rho \int_0^x \pi y^2 \mathrm{d}x + P = \sigma_a \pi y^2.$$

Differentiating with respect to x yields

$$\rho \pi y^2 = \sigma_a \pi \cdot 2y \frac{\mathrm{d}y}{\mathrm{d}x}. \qquad \text{✍ Variable separable}$$

Example 4.25

Evaluate $y_P = \dfrac{1}{(D-2)^3}\,e^{2x}.$

Use Theorem 4:

$$\phi(D) = (D-2)^3, \qquad \phi(2) = 0,$$
$$\phi'(D) = 3(D-2)^2, \qquad \phi'(2) = 0,$$
$$\phi''(D) = 6(D-2), \qquad \phi''(2) = 0,$$
$$\phi'''(D) = 6, \qquad \phi'''(2) = 6 \neq 0.$$
$$\therefore \quad y_P = \frac{1}{\phi'''(2)}\,x^3 e^{2x} = \tfrac{1}{6}\,x^3 e^{2x}.$$

Example 4.26

Solve $(D^2 + 4D + 13)y = e^{-2x}\sin 3x.$

The characteristic equation is $\lambda^2+4\lambda+13=0$, which gives

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4\times 13}}{2} = -2 \pm i3.$$

Hence the complementary solution is $y_C = e^{-2x}(A\cos 3x + B\sin 3x)$.

Remarks: Note that the right-hand side of the differential equation is contained in the complementary solution. Using the method of undetermined coefficient, the assumed form of a particular solution is $x\cdot e^{-2x}(a\cos 3x + b\sin 3x)$.

A particular solution is given by

$$y_P = \frac{1}{D^2+4D+13}\big(e^{-2x}\sin 3x\big) = e^{-2x}\,\frac{1}{(D-2)^2+4(D-2)+13}\,\sin 3x$$

✍ Theorem 2: take e^{-2x} out of the operator, shift D by -2.

$$= e^{-2x}\,\frac{1}{D^2+9}\,\sin 3x = e^{-2x}\,\mathcal{Im}\Big[\frac{1}{D^2+9}\,e^{i3x}\Big].$$

This can be evaluated using Theorem 4:

$$\phi(D) = D^2+9, \qquad \phi(i3) = (i3)^2+9 = 0,$$
$$\phi'(D) = 2D, \qquad \phi'(i3) = 2(i3) = i6 \neq 0.$$

Hence,

$$y_P = e^{-2x}\,\mathcal{Im}\Big[\frac{1}{\phi'(i3)}\,x\,e^{i3x}\Big] \qquad \text{✍ Theorem 4}$$
$$= e^{-2x}\,\mathcal{Im}\Big[\frac{1}{i6}\,x(\cos 3x + i\sin 3x)\Big] = e^{-2x}\,\mathcal{Im}\Big[-\frac{i}{6}\,x(\cos 3x + i\sin 3x)\Big]$$
$$= -\tfrac{1}{6}\,x e^{-2x}\cos 3x.$$

5.2 Electric Circuits

Series RLC Circuit

A circuit consisting of a resistor R, an inductor L, a capacitor C, and a voltage source $V(t)$ connected in series, shown in Figure 5.18, is called the series RLC circuit. Applying Kirchhoff's Voltage Law, one has

$$-V(t) + Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int_{-\infty}^{t} i\,\mathrm{d}t = 0.$$

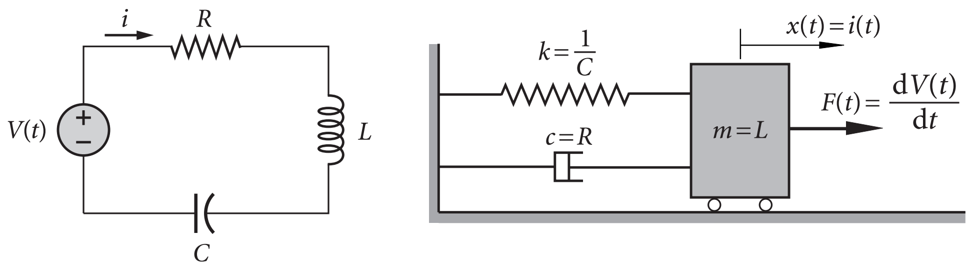


Figure 5.18 Series RLC circuit.

Differentiating with respect to t yields

$$L\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + R\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}i = \frac{\mathrm{d}V(t)}{\mathrm{d}t},$$

or, in the standard form,

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + 2\zeta\omega_0\frac{\mathrm{d}i}{\mathrm{d}t} + \omega_0^2 i = \frac{1}{L}\frac{\mathrm{d}V(t)}{\mathrm{d}t}, \quad \omega_0^2 = \frac{1}{LC}, \quad \zeta\omega_0 = \frac{R}{2L}.$$

The series RLC circuit is equivalent to a mass-damper-spring system as shown.

Parallel RLC Circuit

A circuit consisting of a resistor R, an inductor L, a capacitor C, and a current source $I(t)$ connected in parallel, as shown in Figure 5.19, is called the parallel RLC circuit. Applying Kirchhoff's Current Law at node 1, one has

$$I(t) = C\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{L}\int_{-\infty}^{t} v\,\mathrm{d}t + \frac{v}{R}.$$

Differentiating with respect to t yields

$$C\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \frac{1}{R}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{L}v = \frac{\mathrm{d}I(t)}{\mathrm{d}t},$$

or, in the standard form,

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + 2\zeta\omega_0\frac{\mathrm{d}v}{\mathrm{d}t} + \omega_0^2 v = \frac{1}{C}\frac{\mathrm{d}I(t)}{\mathrm{d}t}, \quad \omega_0^2 = \frac{1}{LC}, \quad \zeta\omega_0 = \frac{1}{2RC}.$$

The parallel RLC circuit is equivalent to a mass-damper-spring system as shown.

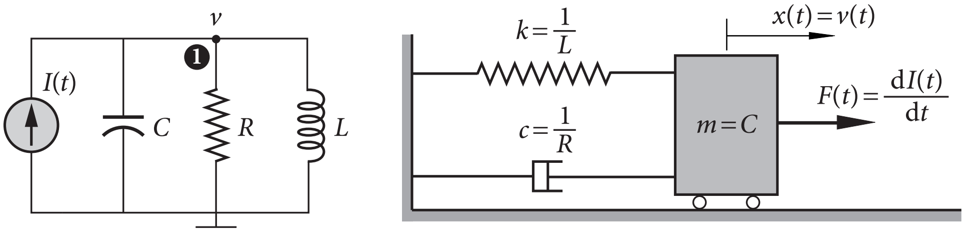
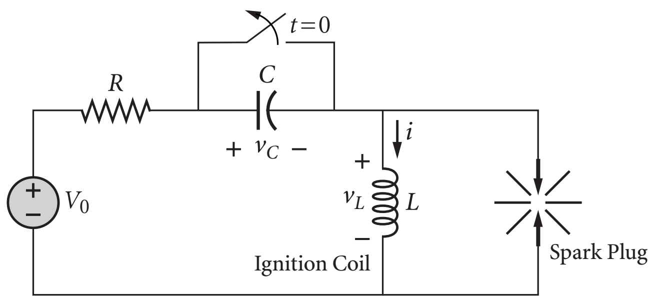


Figure 5.19 Parallel RLC circuit.

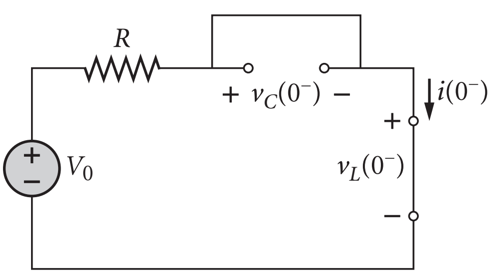
Example 5.1 — Automobile Ignition Circuit

An automobile ignition system is modeled by the circuit shown in the following figure. The voltage source V_0 represents the battery and alternator. The resistor R models the resistance of the wiring, and the ignition coil is modeled by the inductor L. The capacitor C, known as the condenser, is in parallel with the switch, which is known as the electronic ignition. The switch has been closed for a long time prior to $t < 0^-$. Determine the inductor voltage v_L for $t > 0$.

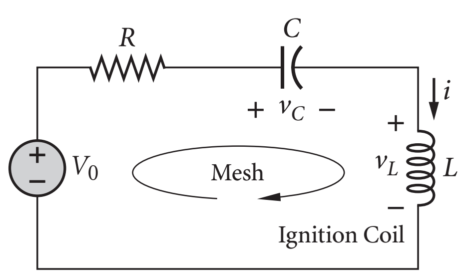


For $V_0 = 12$ V, $R = 4\,\Omega$, $C = 1\,\mu$F, $L = 8\,$mH, determine the maximal inductor voltage and the time when it is reached.

❧ For $t < 0$, the switch is closed, the capacitor behaves as an open circuit and the inductor behaves as a short circuit as shown. Hence $i(0^-) = V_0/R$, $v_C(0^-) = 0$.



$t \leqslant 0^-$



$t \geqslant 0^+$

❧ At $t = 0$, the switch is opened. Since the current in an inductor and the voltage across a capacitor cannot change abruptly, one has $i(0^+) = i(0^-) = V_0/R$, $v_C(0^+) = v_C(0^-) = 0$. The derivative $i'(0^+)$ is obtained from $v_L(0^+)$, which is determined by

5.5 Various Application Problems

Example 5.5 — Jet Engine Vibration

As shown in Figure 5.8, jet engines are supported by the wings of the airplane. To study the horizontal motion of a jet engine, it is modeled as a rigid body supported by an elastic beam. The mass of the engine is m and the moment of inertia about its centroidal axis C is J. The elastic beam is further modeled as a massless bar hinged at A, with the rotational spring κ providing restoring moment equal to $\kappa\theta$, where θ is the angle between the bar and the vertical line as shown in Figure 5.25.

For small rotations, i.e., $|\theta| \ll 1$, set up the equation of motion for the jet engine in term of θ. Find the natural frequency of oscillation.

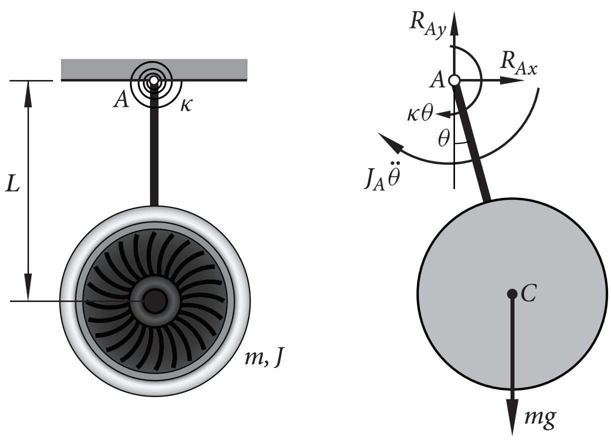


Figure 5.25 Horizontal vibration of a jet engine.

The system rotates about hinge A. The moment of inertia of the jet engine about its centroidal axis C is J. Using the Parallel Axis Theorem, the moment of inertia of the jet engine about axis A is

$$J_A = J + mL^2.$$

Draw the free-body diagram of the jet engine and the supporting bar as shown. The jet engine is subjected to gravity mg. Remove the hinge at A and replace it by two reaction force components R_{Ax} and R_{Ay}. Since the bar rotates an angle θ counterclockwise, the rotational spring provides a clockwise restoring moment $\kappa\theta$.

Since the angular acceleration of the system is $\ddot{\theta}$ counterclockwise, the inertia moment is $J_A\ddot{\theta}$ clockwise.

Applying D'Alembert's Principle, the free-body as shown in Figure 5.25 is in dynamic equilibrium. Hence,

$$\curvearrowright \; \sum M_A = 0: \quad J_A\ddot{\theta} + \kappa\theta + mg \cdot L\sin\theta = 0.$$

Example 5.7 — Single Degree-of-Freedom System

The single degree-of-freedom system described by $x(t)$, as shown in Figure 5.26(a), is subjected to a sinusoidal load $F(t) = F_0 \sin \Omega t$. Assume that the mass m, the spring stiffnesses k_1 and k_2, the damping coefficient c, and F_0 and Ω are known. Determine the *steady-state amplitude* of the response of $x_P(t)$.

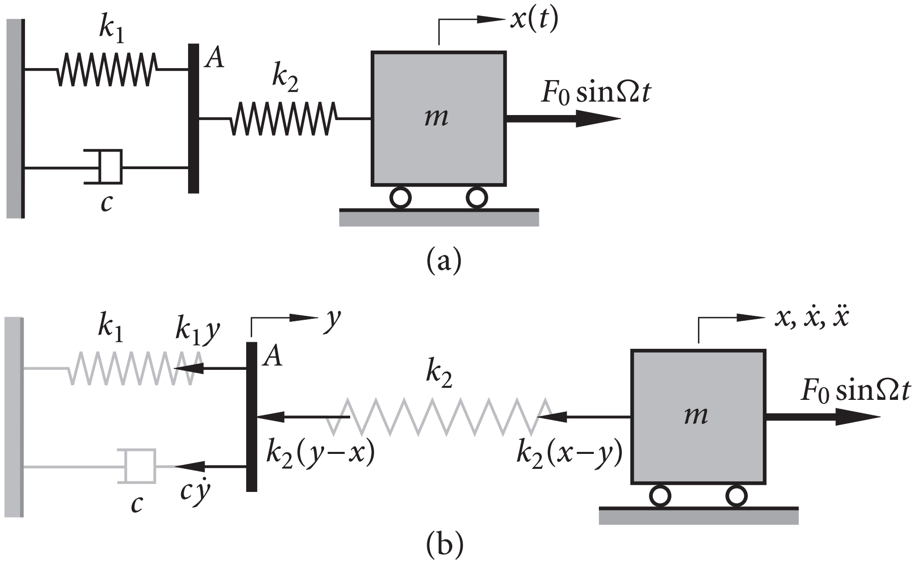


Figure 5.26 A vibrating system.

Introduce a displacement $y(t)$ at A as shown in Figure 5.26(b). Consider the free-body of A. The extension of spring k_1 is y and the compression of spring k_2 is $y-x$. Body A is subjected to three forces: spring force $k_1 y$, damping force $c\dot{y}$, and spring force $k_2(y-x)$. Newton's Second Law requires

$$\rightarrow \; m_A \ddot{y} = \sum F: \qquad m_A \ddot{y} = -k_1 y - c\dot{y} - k_2(y-x).$$

Since the mass of A is zero, i.e., $m_A = 0$, one has

$$x = \frac{(k_1+k_2)y + c\dot{y}}{k_2}. \tag{1}$$

Consider the free-body of mass m. The extension of spring k_2 is $x-y$. The mass is subjected to two forces: spring force $k_2(x-y)$ and the externally applied load $F_0 \sin \Omega t$. Applying Newton's Second Law gives

$$\rightarrow \; m\ddot{x} = \sum F: \qquad m\ddot{x} = F_0 \sin \Omega t - k_2(x-y).$$

Substituting equation (1) yields the equation of motion

$$m\,\frac{(k_1+k_2)\,\ddot{y} + c\,\dddot{y}}{k_2} = F_0 \sin \Omega t - k_2\Big[\frac{(k_1+k_2)y + c\dot{y}}{k_2} - y\Big],$$

Example 6.18

Solve $y'''-y''+4y'-4y=40(t^2+t+1)H(t-2)$, $y(0)=5$, $y'(0)=0$, $y''(0)=10$.

Let $\Upsilon(s)=\mathscr{L}\{y(t)\}$. Taking the Laplace transform of both sides of the differential equation yields

$$\left[s^3\Upsilon(s) - s^2y(0) - sy'(0) - y''(0)\right] - \left[s^2\Upsilon(s) - sy(0) - y'(0)\right] \\ + 4\left[s\Upsilon(s) - y(0)\right] - 4\Upsilon(s) = \mathscr{L}\{40(t^2+t+1)H(t-2)\},$$

where, using $\mathscr{L}\{f(t-a)H(t-a)\}=\mathrm{e}^{-as}\mathscr{L}\{f(t)\}$,

$$\mathscr{L}\{40(t^2+t+1)H(t-2)\} = 40\mathscr{L}\{[(t^2-4t+4)+5t-3]H(t-2)\}$$
$$= 40\mathscr{L}\{[(t-2)^2+5(t-2)+7]H(t-2)\}$$
$$= 40\mathrm{e}^{-2s}\mathscr{L}\{t^2+5t+7\} = 40\mathrm{e}^{-2s}\left(\frac{2!}{s^3}+5\cdot\frac{1!}{s^2}+7\cdot\frac{1}{s}\right) \quad ✍ \ \mathscr{L}\{t^n\}=\frac{n!}{s^{n+1}}$$
$$= \mathrm{e}^{-2s}\,\frac{40(7s^2+5s+2)}{s^3}.$$

Solving for $\Upsilon(s)$ gives

$$\Upsilon(s) = \frac{5s^2-5s+30}{s^3-s^2+4s-4} + \mathrm{e}^{-2s}\,\frac{40(7s^2+5s+2)}{s^3(s^3-s^2+4s-4)}.$$

Using partial fractions, one has

$$\frac{5s^2-5s+30}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4} = \frac{(A+B)s^2+(-B+C)s+(4A-C)}{(s-1)(s^2+4)}$$

To find A, cover-up $(s-1)$ and set $s=1$

$$A = \left.\frac{5s^2-5s+30}{(s^2+4)}\right|_{s=1} = \frac{5-5+30}{1+4} = 6.$$

Comparing the coefficients of the numerators leads to

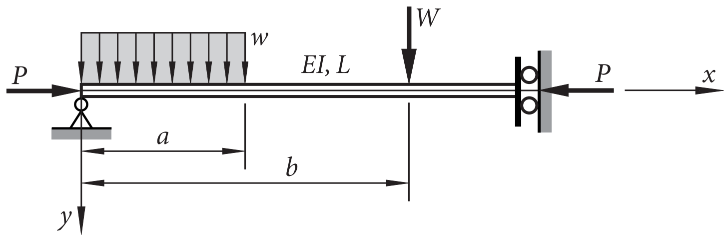
$$s^2:\quad A+B=5 \implies B=5-A=5-6=-1,$$
$$s:\quad -B+C=-5 \implies C=B-5=-1-5=-6,$$
$$1:\quad 4A-C=30.$$ ✍ Use this equation as a check: $4\cdot 6-(-6)=30$.

Hence,

$$\mathscr{L}^{-1}\left\{\frac{5s^2-5s+30}{(s-1)(s^2+4)}\right\} = \mathscr{L}^{-1}\left\{\frac{6}{s-1}+\frac{-s-6}{s^2+4}\right\}$$
$$= \mathscr{L}^{-1}\left\{6\cdot\frac{1}{s-1} - \frac{s}{s^2+2^2} - 3\cdot\frac{2}{s^2+2^2}\right\} = 6\mathrm{e}^t - \cos 2t - 3\sin 2t.$$

Example 6.23 — Beam-Column

Consider the beam-column shown in the following figure. Determine the lateral deflection $y(x)$.



Using the Heaviside step function and the Dirac delta function, the lateral load can be expresses as

$$w(x) = w\big[1 - H(x-a)\big] + W\,\delta(x-b).$$

Following the formulation in Section 5.4, the differential equation becomes

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + \alpha^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \hat{w}\big[1 - H(x-a)\big] + \hat{W}\,\delta(x-b), \quad \alpha^2 = \frac{P}{EI}, \quad \hat{w} = \frac{w}{EI}, \quad \hat{W} = \frac{W}{EI}.$$

Since the left end is a hinge support and the right end is a sliding support, the boundary conditions are

$$\begin{aligned}
\text{at } x = 0: \qquad & \text{deflection} = 0 \implies y(0) = 0,\\
& \text{bending moment} = 0 \implies y''(0) = 0,\\
\text{at } x = L: \qquad & \text{slope} = 0 \implies y'(L) = 0,\\
& \text{shear force} = 0 \implies V(L) = -EI\,y'''(L) - P\,y'(L) = 0\\
& \implies y'''(L) = 0.
\end{aligned}$$

Applying the Laplace transform $\Upsilon(s) = \mathscr{L}\{y(x)\}$, one has

$$\big[s^4\,\Upsilon(s) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0)\big] + \alpha^2\big[s^2\,\Upsilon(s) - s\,y(0) - y'(0)\big] = \frac{\hat{w}}{s}\,(1 - \mathrm{e}^{-as}) + \hat{W}\,\mathrm{e}^{-bs}.$$

Since $y(0) = y''(0) = 0$, solving for $\Upsilon(s)$ leads to

$$\Upsilon(s) = \frac{y'(0)}{s^2 + \alpha^2} + \frac{\big[y'''(0) + \alpha^2 y'(0)\big] + \hat{W}\,\mathrm{e}^{-bs}}{s^2(s^2 + \alpha^2)} + \frac{\hat{w}}{s^3(s^2 + \alpha^2)}\,(1 - \mathrm{e}^{-as}).$$

Applying partial fractions

$$\frac{1}{s^3(s^2 + \alpha^2)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{Ds + E}{s^2 + \alpha^2}.$$

The general solution is

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{X}(t)\left\{\mathbf{C}+\int \mathbf{X}^{-1}(t)\,\mathbf{f}(t)\,\mathrm{d}t\right\}\\ &= \begin{bmatrix}\cos t & \sin t\\ \sin t+\cos t & \sin t-\cos t\end{bmatrix}\begin{Bmatrix}C_1+t+\ln|\cos t|\\ C_2+t-\ln|\cos t|\end{Bmatrix},\end{aligned}$$

$$\therefore\quad x_1(t) = (t+C_1)\cos t + (t+C_2)\sin t + (\cos t-\sin t)\ln|\cos t|,$$

$$x_2(t) = (C_1-C_2)\cos t + (2t+C_1+C_2)\sin t + 2\cos t\ln|\cos t|.$$

Example 7.20

Solve

$$\mathbf{x}'(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{f}(t),\quad \mathbf{x}(t)=\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix},\quad \mathbf{A}=\begin{bmatrix}2 & -1 & -1\\ 2 & -1 & -2\\ -1 & 1 & 2\end{bmatrix},\quad \mathbf{f}(t)=\begin{Bmatrix}2\mathrm{e}^t\\ 4\mathrm{e}^{-t}\\ 0\end{Bmatrix}.$$

The characteristic equation is

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix}2-\lambda & -1 & -1\\ 2 & -1-\lambda & -2\\ -1 & 1 & 2-\lambda\end{vmatrix} = -(\lambda^3-3\lambda^2+3\lambda-1) = -(\lambda-1)^3 = 0.$$

Hence, $\lambda=1$ is an eigenvector of multiplicity 3. The eigenvector equation is

$$(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{v} = \begin{bmatrix}1 & -1 & -1\\ 2 & -2 & -2\\ -1 & 1 & 1\end{bmatrix}\begin{Bmatrix}v_1\\ v_2\\ v_3\end{Bmatrix} = \begin{Bmatrix}v_1-v_2-v_3\\ 2(v_1-v_2-v_3)\\ -(v_1-v_2-v_3)\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix},$$

which leads to $v_1=v_2+v_3$. As a result, there are two linearly independent eigenvectors. Taking $v_{21}=1$ and $v_{31}=-1$, then $v_{11}=v_{21}+v_{31}=0$,

$$\therefore\quad \mathbf{v}_1 = \begin{Bmatrix}v_{11}\\ v_{21}\\ v_{31}\end{Bmatrix} = \begin{Bmatrix}0\\ 1\\ -1\end{Bmatrix}.$$

However, $\mathbf{v}_2$ cannot be chosen arbitrarily; it has to satisfy a condition imposed by $\mathbf{v}_3$, which will be clear in a moment.

A third linearly independent eigenvector does not exist. Hence, matrix $\mathbf{A}$ is defective and a complete basis of eigenvectors is obtained by including one generalized eigenvector:

$$(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{v}_3=\mathbf{v}_2 \implies \begin{bmatrix}1 & -1 & -1\\ 2 & -2 & -2\\ -1 & 1 & 1\end{bmatrix}\begin{Bmatrix}v_{13}\\ v_{23}\\ v_{33}\end{Bmatrix} = \begin{Bmatrix}v_{13}-v_{23}-v_{33}\\ 2(v_{13}-v_{23}-v_{33})\\ -(v_{13}-v_{23}-v_{33})\end{Bmatrix} = \begin{Bmatrix}v_{12}\\ v_{22}\\ v_{32}\end{Bmatrix}.$$

8.2 Vibration Absorbers or Tuned Mass Dampers

In engineering applications, many systems can be modeled as single degree-of-freedom systems. For example, a machine mounted on a structure can be modeled using a mass-spring-damper system, in which the machine is considered to be rigid with mass m and the supporting structure is equivalent to a spring k and a damper c, as shown in Figure 8.2. The machine is subjected to a sinusoidal force $F_0 \sin \Omega t$, which can be an externally applied load or due to imbalance in the machine.

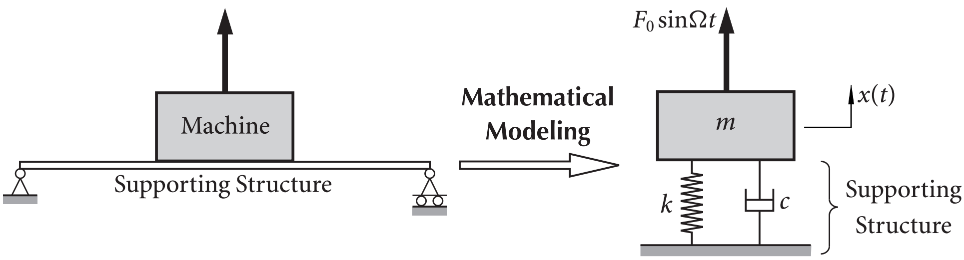


Figure 8.2 A machine mounted on a structure.

From Chapter 5 on the response of a single degree-of-freedom system, it is well known that when the excitation frequency Ω is close to the natural frequency of the system $\omega_0 = \sqrt{k/m}$, vibration of large amplitude occurs. In particular, when the system is undamped, i.e., $c=0$, resonance occurs when $\Omega = \omega_0$, in which the amplitude of the response grows linearly with time.

To reduce the vibration of the system, a vibration absorber or a tuned mass damper (TMD), which is an auxiliary mass-spring-damper system, is mounted on the main system as shown in Figure 8.3(a). The mass, spring stiffness, and damping coefficient of the viscous damper are m_a, k_a, and c_a, respectively, where the subscript “a” stands for “auxiliary.”

To derive the equation of motion of the main mass m, consider its free-body diagram as shown in Figure 8.3(b). Since mass m moves upward, spring k is extended and spring k_a is compressed.

- Because of the displacement x of mass m, the extension of spring k is x. Hence the spring k exerts a downward force kx and the damper c exerts a downward force $c\dot{x}$ on mass m.
- Because the mass m_a also moves upward a distance x_a, the net compression in spring k_a is $x - x_a$. Hence the spring k_a and damper c_a exert downward forces $k_a(x - x_a)$ and $c_a(\dot{x} - \dot{x}_a)$, respectively, on mass m.

Newton's Second Law requires

$$\uparrow \; m\ddot{x} = \sum F: \quad m\ddot{x} = -kx - c\dot{x} - k_a(x - x_a) - c_a(\dot{x} - \dot{x}_a) + F_0 \sin \Omega t,$$

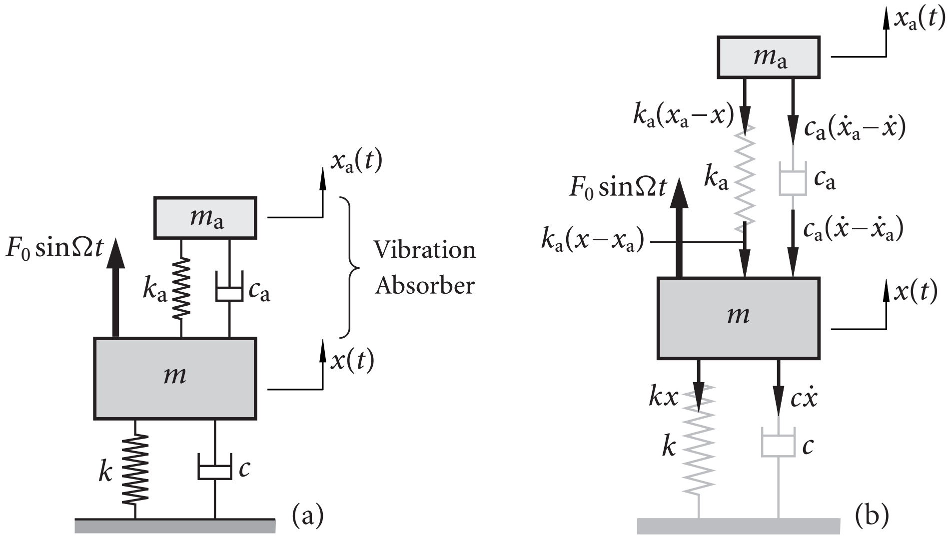


Figure 8.3 A vibration absorber mounted on the main system.

or

$$m\ddot{x} + (c+c_a)\dot{x} + (k+k_a)x - c_a\dot{x}_a - k_a x_a = F_0 \sin\Omega t.$$

Similarly, consider the free-body diagram of mass m_a. Since mass m_a moves upward a distance $x_a(t)$, spring k_a is extended. The net extension of spring k_a is $x_a - x$. Hence, the spring k_a and damper c_a exert downward forces $k_a(x_a - x)$ and $c_a(\dot{x}_a - \dot{x})$, respectively. Applying Newton's Second Law gives

$$\uparrow\ m_a\ddot{x}_a = \sum F: \quad m_a\ddot{x}_a = -k_a(x_a - x) - c_a(\dot{x}_a - \dot{x}),$$

$$\therefore \quad m_a\ddot{x}_a + c_a\dot{x}_a + k_a x_a - c_a\dot{x} - k_a x = 0.$$

The equations of motion can be written using the $\mathcal{D}$-operator as

$$\big[m\mathcal{D}^2 + (c+c_a)\mathcal{D} + (k+k_a)\big]x - (c_a\mathcal{D} + k_a)x_a = F_0 \sin\Omega t,$$

$$-(c_a\mathcal{D} + k_a)x + (m_a\mathcal{D}^2 + c_a\mathcal{D} + k_a)x_a = 0.$$

Because of the existence of damping, the responses of free vibration (complementary solutions) decay exponentially and approach zero as time increases. Hence, it is practically more important and useful to study responses of *forced vibration* (*particular solutions*). The determinant of the coefficient matrix is

$$\phi(\mathcal{D}) = \begin{vmatrix} m\mathcal{D}^2 + (c+c_a)\mathcal{D} + (k+k_a) & -(c_a\mathcal{D} + k_a) \\ -(c_a\mathcal{D} + k_a) & m_a\mathcal{D}^2 + c_a\mathcal{D} + k_a \end{vmatrix}$$

$$= \big[m\mathcal{D}^2 + (c+c_a)\mathcal{D} + (k+k_a)\big](m_a\mathcal{D}^2 + c_a\mathcal{D} + k_a) - (c_a\mathcal{D} + k_a)^2$$

$$= \big[(m\mathcal{D}^2 + k)(m_a\mathcal{D}^2 + k_a) + k_a m_a\mathcal{D}^2 + c_a c\,\mathcal{D}^2\big]$$

$$+ \big[c_a(m\mathcal{D}^2 + k) + c(m_a\mathcal{D}^2 + k_a) + c_a m_a\mathcal{D}^2\big]\mathcal{D},$$

Example 9.8

Obtain series solution about $x=0$ of the equation

$$2x^2y'' + x(2x+1)y' - y = 0.$$

The differential equation is of the form

$$y'' + P(x)y' + Q(x)y = 0, \quad P(x) = \frac{2x+1}{2x}, \quad Q(x) = -\frac{1}{2x^2}.$$

Obviously, $x=0$ is a singular point. Note that

$$xP(x) = \frac{2x+1}{2} = \frac{1}{2} + x + 0\cdot x^2 + 0\cdot x^3 + \cdots \implies P_0 = \frac{1}{2},$$

$$x^2Q(x) = -\frac{1}{2} = -\frac{1}{2} + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 + \cdots \implies Q_0 = -\frac{1}{2}.$$

Both $xP(x)$ and $x^2Q(x)$ are analytic at $x=0$ and can be expanded as power series that are convergent for $|x|<\infty$. Hence, $x=0$ is a regular singular point.

The indicial equation is $\alpha(\alpha-1)+\alpha P_0+Q_0=0$:

$$\alpha(\alpha-1) + \alpha\cdot\frac{1}{2} - \frac{1}{2} = 0 \implies (\alpha+\tfrac{1}{2})(\alpha-1) = 0 \implies \alpha_1=1, \ \alpha_2=-\frac{1}{2}.$$

Thus the equation has a Frobenius series solution of the form

$$y_1(x) = x^{\alpha_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}, \quad a_0\neq 0, \quad 0<x<\infty,$$

where a_n, $n=0,1,\ldots,$ are constants to be determined. Differentiating with respect to x yields

$$y_1'(x) = \sum_{n=0}^{\infty}(n+1)a_n x^n, \quad y_1''(x) = \sum_{n=1}^{\infty}(n+1)na_n x^{n-1}.$$

Substituting y_1, y_1', and y_1'' into the differential equation results in

$$\sum_{n=1}^{\infty}2(n+1)na_nx^{n+1} + \sum_{n=0}^{\infty}2(n+1)a_nx^{n+2} + \sum_{n=0}^{\infty}(n+1)a_nx^{n+1} - \sum_{n=0}^{\infty}a_nx^{n+1} = 0.$$

Changing the indices of the summations

$$\sum_{n=1}^{\infty}2(n+1)na_nx^{n+1} \xrightarrow{n+1=m} \sum_{m=2}^{\infty}2m(m-1)a_{m-1}x^m,$$

$$\sum_{n=0}^{\infty}2(n+1)a_nx^{n+2} \xrightarrow{n+2=m} \sum_{m=2}^{\infty}2(m-1)a_{m-2}x^m,$$

$$\sum_{n=0}^{\infty}na_nx^{n+1} \xrightarrow{n+1=m} \sum_{m=1}^{\infty}(m-1)a_{m-1}x^m,$$

one obtains

$$\sum_{n=2}^{\infty}\left[2n(n-1)\,a_{n-1}+2(n-1)\,a_{n-2}\right]x^n+\sum_{n=1}^{\infty}(n-1)\,a_{n-1}\,x^n=0.$$

For this equation to be true, the coefficient of x^n, $n=1,2,\ldots$, must be zero. For $n=1$, one has

$$0\cdot a_0=0 \quad\Longrightarrow\quad a_0\neq 0 \text{ is arbitrary; take } a_0=1.$$

For $n\geqslant 2$, one has

$$2n(n-1)\,a_{n-1}+2(n-1)\,a_{n-2}+(n-1)\,a_{n-1}=0 \quad\Longrightarrow\quad a_{n-1}=-\frac{2a_{n-2}}{2n+1}.$$

Hence,

$$\begin{aligned}
n=2:&\quad a_1=-\frac{2a_0}{2\cdot 2+1}=-\frac{2}{5},\\
n=3:&\quad a_2=-\frac{2a_1}{2\cdot 3+1}=(-1)^2\frac{2^2}{7\cdot 5},\\
&\ \ \vdots\\
n+1:&\quad a_n=-\frac{2a_{n-1}}{2(n+1)+1}=(-1)^n\frac{2^n}{(2n+3)(2n+1)\cdots 5}=(-1)^n\frac{3\cdot 2^n}{(2n+3)!!},
\end{aligned}$$

where $(2n+3)!!=(2n+3)(2n+1)\cdots 5\cdot 3\cdot 1$ is the double factorial. The first Frobenius series solution is

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 2^n}{(2n+3)!!}x^{n+1},\quad 0<x<\infty.$$

Since $\alpha_1-\alpha_2=\frac{3}{2}$, according to Fuchs' Theorem, a second linearly independent solution is also a Frobenius series given by

$$y_2(x)=x^{\alpha_2}\sum_{n=0}^{\infty}b_n x^n=\sum_{n=0}^{\infty}b_n x^{n-\frac{1}{2}},\quad b_0\neq 0,\quad 0<x<\infty,$$

$$y_2'(x)=\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)b_n x^{n-\frac{3}{2}},\quad y_2''(x)=\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)b_n x^{n-\frac{5}{2}}.$$

Substituting y_2, y_2', and y_2'' into the differential equation leads to

$$2x^2\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)b_n x^{n-\frac{5}{2}}+(2x^2+x)\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)b_n x^{n-\frac{3}{2}}-\sum_{n=0}^{\infty}b_n x^{n-\frac{1}{2}}=0,$$

$$\sum_{n=0}^{\infty}\left\{\left[2\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)+\left(n-\tfrac{1}{2}\right)-1\right]b_n x^{n-\frac{1}{2}}+2\left(n-\tfrac{1}{2}\right)b_n x^{n+\frac{1}{2}}\right\}=0.$$

A schematic diagram is shown in Figure 10.3 to illustrate the procedure of the improved Euler predictor-corrector method.

Improved Euler Predictor-Corrector Method

At the $(i+1)$th step, $i=0, 1, 2, \ldots,$

(1)	$k_1 = f(x_i, y_i)$	Slope at the left end point x_i
(2) **Predictor**	$y^{P}_{i+1} = y_i + hk_1$	Predict y at x_{i+1} using the Euler method
(3)	$k_2 = f(x_{i+1}, y^{P}_{i+1})$	Predicted slope at the right end point x_{i+1}
(4)	$k = \dfrac{k_1 + k_2}{2}$	The averaged slope is used on $[x_i, x_{i+1}]$
(5) **Corrector**	$y_{i+1} = y_i + hk$	Improved Euler point

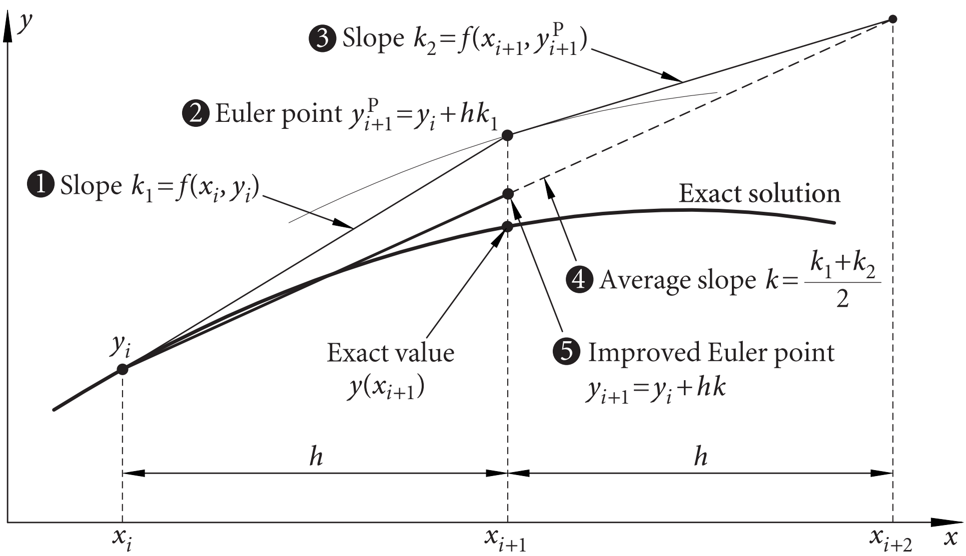


Figure 10.3 Improved Euler predictor-corrector method.

Example 10.3

For the initial value problem $y'=xy^2-y$, $y(0)=0.5$, determine $y(1.0)$ using the improved Euler method and the improved Euler predictor-corrector method with $h=0.5$.

(1) The improved Euler method is, with $f(x, y)=xy^2-y$ and $h=0.5$,

$$y_{i+1} = y_i + \tfrac{1}{2}h\big[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\big].$$

The results are as follows

$i=0$: $x_0=0$, $y_0=0.5$,

11.4.3 One-Dimensional Transient Heat Conduction

Consider a wall or plate of infinite size and of thickness L, as shown in Figure 11.7, which is suddenly exposed to fluids in motion on both of its surfaces. The coefficient of thermal conductivity of the wall or plate is k. Suppose the wall has an initial temperature distribution $T(x,0)=f(x)$. The temperatures of the fluids and the heat transfer coefficients on the left-hand and right-hand sides of the wall are T_{f1}, h_1 and T_{f2}, h_2, respectively.

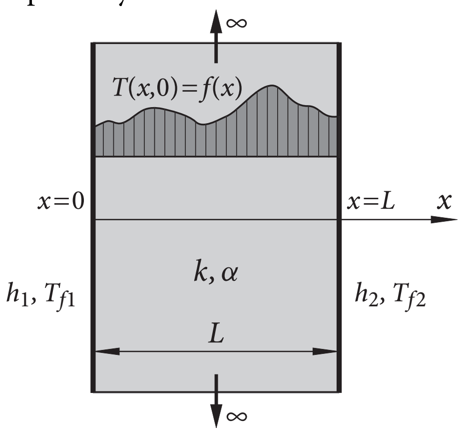


Figure 11.7 An infinite wall.

Because the wall or plate is infinitely large, the heat transfer process is simplified as one-dimensional (in the x-dimension).

The differential equation (Fourier's equation in one-dimension), the initial condition, and the boundary conditions of this one-dimensional transient heat conduction problem are

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad 0 \leqslant x \leqslant L, \quad t \geqslant 0,$$

Initial Condition (IC): $$T = f(x), \qquad \text{at } t=0,$$

Boundary Conditions (BCs): $$k\frac{\partial T}{\partial x} = h_1(T - T_{f1}), \qquad \text{at } x=0,$$

$$-k\frac{\partial T}{\partial x} = h_2(T - T_{f2}), \qquad \text{at } x=L.$$

This mathematical model has many engineering applications.

- The infinite wall is a model of a flat wall of a heat exchanger, which is initially isothermal at $T=T_0$. The operation of the heat exchanger is initiated at $t=0$; two different fluids of temperatures T_{f1} and T_{f2}, respectively, are flowing along the sides of the wall.
- The infinite wall is a model of a wall in a building or a furnace. One side of the wall is suddenly exposed to a higher temperature T_{f1} due to fire occurring in a room or the ignition of flames in the furnace.

Example 12.23 — Dynamical Response of Parametrically Excited System

Consider the parametrically excited nonlinear system given by

$$\ddot{x} + \beta\,\dot{x} - (1 + \mu\cos\Omega t)\,x + \alpha\,x^3 = 0.$$

Examples of this equation are found in many applications of mechanics, especially in problems of dynamic stability of elastic systems. In particular, the transverse vibration of a buckled column under the excitation of a periodic end displacement is described by this equation. The system is called parametrically excited because the forcing term $\mu\cos\Omega t$ appears in the coefficient (parameter) of the equation.

✍ It is a good practice to put `restart` at the beginning of each program so that *Maple* can start fresh if the program has to be rerun.

```
>restart:
>with(plots):
```
✍ Load the `plots` package.

```
>ODE:=diff(x(t),t$2)+beta*diff(x(t),t)-(1+mu*cos(Omega*t))*x(t)
 +alpha*x(t)^3=0:
```
✍ Define the ODE.

```
>ICs:=x(0)=0,D(x)(0)=0.1:
```
✍ Define the ICs: $x(0)=0$, $\dot{x}(0)=0.1$.

Periodic Motion ($\mu = 0.3$)

```
>alpha:=1.0: beta:=0.2: Omega:=1.0: mu:=0.3:
```
✍ Assign the parameters.

✍ Solve the system numerically using `dsolve` with option `numeric`.

```
>sol:=dsolve({ODE,ICs},x(t),numeric,maxfun=1000000):
```

✍ Plot the time series $x(t)$ versus t, $(x_1 = x)$. ✍ Figure 12.1(a)

```
>odeplot(sol,[t,x(t)],t=0..500,numpoints=10000,labels=["t","x1"],
 tickmarks=[[0,100,200,300,400,500],[-1.5,-1,-0.5,0,0.5,1,1.5]]);
```

✍ Plot the time series $\dot{x}(t)$ versus t, $(x_2 = \dot{x})$. ✍ Figure 12.1(b)

```
>odeplot(sol,[t,D(x)(t)],t=0..500,numpoints=10000,labels=["t","x2"],
 tickmarks=[[0,100,200,300,400,500],[-0.8,-0.6,-0.4,-0.2,0,0.2,0.4,
 0.6,0.8]]);
```

✍ Plot the phase portrait $\dot{x}(t)$ versus $x(t)$. ✍ Figure 12.1(c)

```
>odeplot(sol,[x(t),D(x)(t)],t=0..500,numpoints=10000,view=[-1.8..1.8,
 -1.0..1.0], tickmarks=[[-1.8,-1.2,-0.6,0.6,1.2,1.8],[-1,-0.75,-0.5,
 -0.25,0.25,0.5,0.75,1]],axes=normal,labels=["x1","x2"]);
```

✍ When $\mu = 0.3$, after some transient part, the response of the system will settle down to periodic motion.

Chaotic Motion ($\mu = 0.4$)

```
>alpha:=1.0: beta:=0.2: Omega:=1.0: mu:=0.4:
```
✍ Assign the parameters.

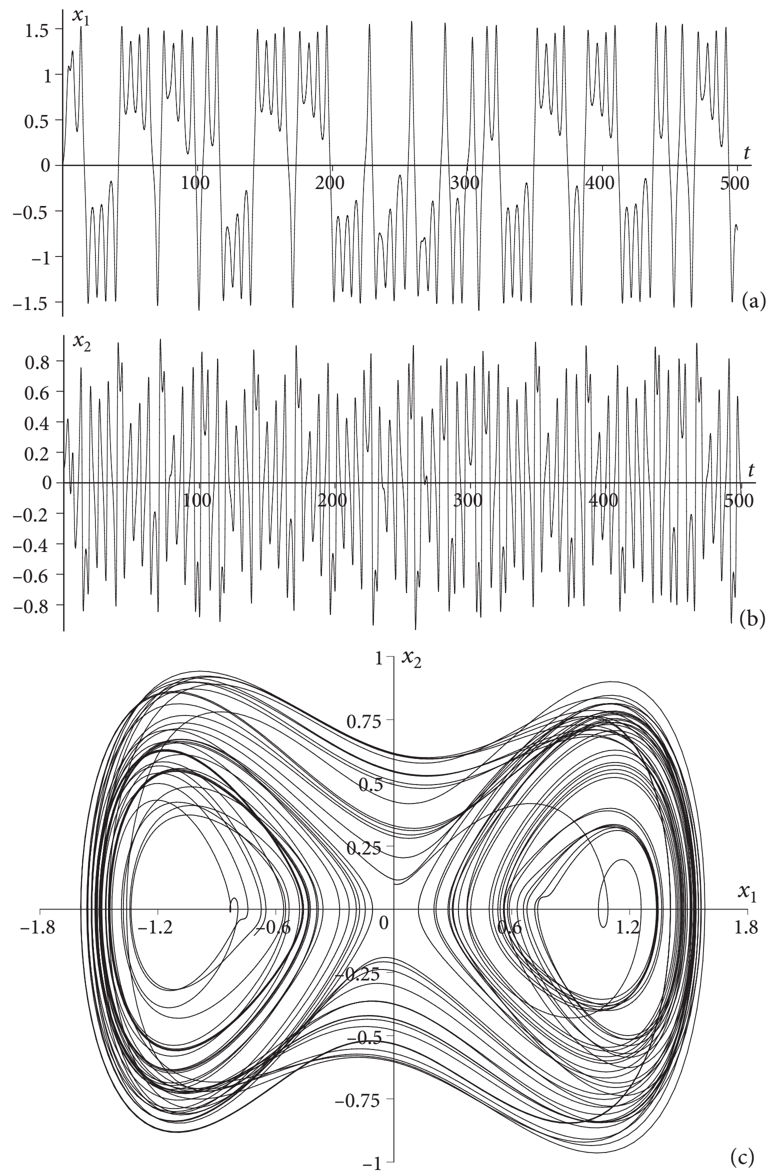


Figure 12.2 Chaotic motion.

✍ Plot the time series $\dot{x}(t)$ versus t. ✍ Figure 12.2(b)

```
>odeplot(sol,[t,D(x)(t)],t=0..500,numpoints=10000,labels=["t","x2"],
 tickmarks=[[0,100,200,300,400,500],[-0.8,-0.6,-0.4,-0.2,0,0.2,0.4,
 0.6,0.8]]);
```

✍ Plot the phase portrait $\dot{x}(t)$ versus $x(t)$. ✍ Figure 12.2(c)

```
>odeplot(sol,[x(t),D(x)(t)],t=0..500,numpoints=10000,view=[-1.8..1.8,
 -1.0..1.0], tickmarks=[[-1.8,-1.2,-0.6,0.6,1.2,1.8],[-1,-0.75,-0.5,
 -0.25,0.25,0.5,0.75,1]],axes=normal,labels=["x1","x2"]);
```

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