## DIFFERENTIAL EQUATIONS FOR ENGINEERS

This book presents a systematic and comprehensive introduction to ordinary differential equations for engineering students and practitioners. Mathematical concepts and various techniques are presented in a clear, logical, and concise manner. Various visual features are used to highlight focus areas. Complete illustrative diagrams are used to facilitate mathematical modeling of application problems. Readers are motivated by a focus on the relevance of differential equations through their applications in various engineering disciplines. Studies of various types of differential equations are determined by engineering applications. Theory and techniques for solving differential equations are then applied to solve practical engineering problems. Detailed step-by-step analysis is presented to model the engineering problems using differential equations from physical principles and to solve the differential equations using the easiest possible method. Such a detailed, step-by-step approach, especially when applied to practical engineering problems, helps the readers to develop problem-solving skills.

This book is suitable for use not only as a textbook on ordinary differential equations for undergraduate students in an engineering program but also as a guide to self-study. It can also be used as a reference after students have completed learning the subject.

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**2.100** $3yy'y'' - y'^3 + 1 = 0$ ANS $3(C_1 y + 1)^{2/3} - 2C_1 x = C_2; \quad y = x$

**2.101** $y'' - y'^2 - 1 = 0$ ANS $y = -\ln\left|\cos(x + C_1)\right| + C_2$

**2.102** $x^3 y'' - x^2 y' = 3 - x^2$ ANS $y = \dfrac{1}{x} + x + C_1 x^2 + C_2$

**2.103** $2y'' = y'^3 \sin 2x, \; y(0) = 1, \; y'(0) = 1$ ANS $y = 1 + \ln\left|\sec x + \tan x\right|$

**2.104** $x\dfrac{d^2 y}{dx^2} = 2 - \dfrac{dy}{dx}$ ANS $y = 2x + C_1 \ln|x| + C_2$

**2.105** $y'' = 3\sqrt{y}, \quad y(0) = 1, \; y'(0) = 2$ ANS $y = \left(\pm \frac{1}{2}x + 1\right)^4$

**2.106** $x\dfrac{d^2 y}{dx^2} = \dfrac{dy}{dx} + x \sin\left(\dfrac{1}{x} \cdot \dfrac{dy}{dx}\right)$

ANS $y = \left(x^2 + \dfrac{1}{C_1^2}\right)\tan^{-1} C_1 x - \dfrac{x}{C_1} + C_2; \quad y = \dfrac{k\pi}{2}x^2 + C, \; k = 0, \pm 1, \pm 2, \ldots$

**2.107** $yy'' = y'^2(1 - y' \sin y - yy' \cos y)$

ANS $y = C; \quad x = -\cos y + C_1 \ln|y| + C_2$

**2.108** $y'' + xy' = x$ ANS $y = x + C_1 \displaystyle\int e^{-\frac{1}{2}x^2} dx + C_2$

**2.109** $xy'' - y'^3 - y' = 0$ ANS $x^2 + (y - C_1)^2 = C_2; \quad y = C$

**2.110** $y(1 - \ln y)y'' + (1 + \ln y)y'^2 = 0$

ANS $y = C; \quad (C_1 x + C_2)(\ln y - 1) + 1 = 0$

## *Review Problems*

**2.111** $xy^2(xy' + y) = 1$ ANS $2x^3y^3 - 3x^2 = C$

**2.112** $5y + y'^2 = x(x + y')$ ANS $4y = x^2; \; 5y = -5x^2 + 5Cx - C^2$

**2.113** $y' = \dfrac{y+2}{x+1} + \tan\dfrac{y-2x}{x+1}$ ANS $\sin\dfrac{y-2x}{x+1} = C(x+1)$

**2.114** $y''(e^x + 1) + y' = 0$ ANS $y = C_1(x - e^{-x}) + C_2$

**2.115** $xy' = y - xe^{y/x}$ ANS $y = -x\ln\Big|\ln|Cx|\Big|$

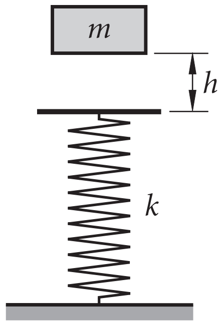
**2.116** $(1 + y^2 \sin 2x)\,dx - 2y\cos^2 x\,dy = 0$ ANS $x - y^2\cos^2 x = C$

**2.117** $(2\sqrt{xy} - y)\,dx - x\,dy = 0, \; x > 0, \; y > 0$ ANS $\sqrt{xy} - x = C$

**2.118** $y'' + y'^2 = 2e^{-y}$ ANS $e^y + C_1 = (x + C_2)^2$

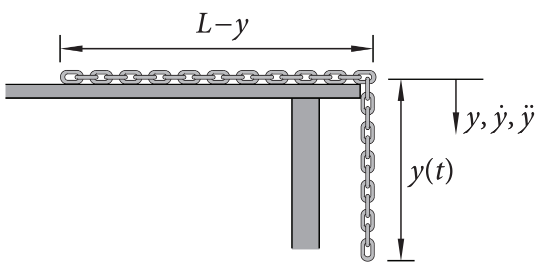
**5.4** A mass $m$ is dropped with zero initial velocity from a height of $h$ above a spring of stiffness $k$ as shown in the following figure. Determine the maximum compression of the spring and the duration between the time when the mass contacts the spring and the time when the spring reaches maximum compression.

ANS $y_{\max} = \sqrt{\frac{mg}{k}\left(2h+\frac{mg}{k}\right)}+\frac{mg}{k}, \quad T = \sqrt{\frac{m}{k}}\left(\frac{\pi}{2}+\tan^{-1}\sqrt{\frac{mg}{2hk}}\right)$




**5.5** A uniform chain of length $L$ with mass density per unit length $\rho$ is laid on a rough horizontal table with an initial hang of length $l$, i.e., $y=l$ at $t=0$ as shown in the following figure. The coefficients of static and kinetic friction between the chain and the surface have the same value $\mu$. The chain is released from rest at time $t=0$ and it starts sliding off the table if $(1+\mu)l>\mu L$. Show that the time $T$ it takes for the chain to leave the table is

$$T = \sqrt{\frac{L}{(1+\mu)g}}\cosh^{-1}\left[\frac{L}{(1+\mu)l-\mu L}\right].$$




**5.6** A uniform chain of length $L$ with mass density per unit length $\rho$ is laid on a smooth inclined surface with $y=0$ at $t=0$ as shown in the following figure. The chain is released from rest at time $t=0$. Show that the time $T$ it takes for the chain to leave the surface is

$$T = \sqrt{\frac{L}{(1-\sin\theta)g}}\cosh^{-1}\left(\frac{1}{\sin\theta}\right).$$

**7.11** $(D-4)x+3y=\sin t, \quad -2x+(D+1)y=-2\cos t$

ANS $x=C_1\mathrm{e}^t+C_2\mathrm{e}^{2t}+\cos t-2\sin t, \;\; y=C_1\mathrm{e}^t+\frac{2}{3}C_2\mathrm{e}^{2t}+2\cos t-2\sin t$

**7.12** $\dfrac{\mathrm{d}x}{\mathrm{d}t}-y=0, \quad -x+\dfrac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^t+\mathrm{e}^{-t}$

ANS $x=C_1\mathrm{e}^t+C_2\mathrm{e}^{-t}+\frac{1}{2}t\mathrm{e}^t-\frac{1}{2}t\mathrm{e}^{-t}$

$y=\left(C_1+\frac{1}{2}\right)\mathrm{e}^t+\left(-C_2-\frac{1}{2}\right)\mathrm{e}^{-t}+\frac{1}{2}t\mathrm{e}^t+\frac{1}{2}t\mathrm{e}^{-t}$

**7.13** $(D+2)x+5y=0, \quad -x+(D-2)y=\sin 2t$

ANS $x=A\cos t+B\sin t+\frac{5}{3}\sin 2t$

$y=-\frac{1}{5}(2A+B)\cos t+\frac{1}{5}(A-2B)\sin t-\frac{2}{3}(\cos 2t+\sin 2t)$

**7.14** $(D-2)x+2Dy=-4\mathrm{e}^{2t}, \quad (2D-3)x+(3D-1)y=0$

ANS $x=C_1\mathrm{e}^{-2t}+C_2\mathrm{e}^t+5\mathrm{e}^{2t}, \quad y=-C_1\mathrm{e}^{-2t}+\frac{1}{2}C_2\mathrm{e}^t-\mathrm{e}^{2t}$

**7.15** $(3D+2)x+(D-6)y=5\mathrm{e}^t, \quad (4D+2)x+(D-8)y=5\mathrm{e}^t+2t-3$

ANS $x=A\cos 2t+B\sin 2t+2\mathrm{e}^t-3t+5, \;\; y=B\cos 2t-A\sin 2t+\mathrm{e}^t-t$

**7.16** $(D-5)x+3y=2\mathrm{e}^{3t}, \quad -x+(D-1)y=5\mathrm{e}^{-t}$

ANS $x=C_1\mathrm{e}^{2t}+3C_2\mathrm{e}^{4t}-\mathrm{e}^{-t}-4\mathrm{e}^{3t}, \; y=C_1\mathrm{e}^{2t}+C_2\mathrm{e}^{4t}-2\mathrm{e}^{-t}-2\mathrm{e}^{3t}$

**7.17** $(D-2)x+y=0, \quad x+(D-2)y=-5\mathrm{e}^t\sin t$

ANS $x=C_1\mathrm{e}^t+C_2\mathrm{e}^{3t}+\mathrm{e}^t(2\cos t-\sin t), \;\; y=C_1\mathrm{e}^t-C_2\mathrm{e}^{3t}+\mathrm{e}^t(3\cos t+\sin t)$

**7.18** $(D+4)x+2y=\dfrac{2}{\mathrm{e}^t-1}, \quad 6x-(D-3)y=\dfrac{3}{\mathrm{e}^t-1}$

ANS $x=C_1+2C_2\mathrm{e}^{-t}+2\mathrm{e}^{-t}\ln\left|\mathrm{e}^t-1\right|, \;\; y=-2C_1-3C_2\mathrm{e}^{-t}-3\mathrm{e}^{-t}\ln\left|\mathrm{e}^t-1\right|$

**7.19** $(D-1)x+y=\sec t, \quad -2x+(D+1)y=0$

ANS $x=C_1\cos t+C_2\sin t+t(\cos t+\sin t)+(\cos t-\sin t)\ln\left|\cos t\right|$

$y=(C_1-C_2)\cos t+(C_1+C_2)\sin t+2t\sin t+2\cos t\ln\left|\cos t\right|$

## *The Method of Laplace Transform*

Solve the following differential equations using the method of Laplace transform.

**7.20** $\dfrac{\mathrm{d}x}{\mathrm{d}t}-x-2y=16t\mathrm{e}^t, \quad 2x-\dfrac{\mathrm{d}y}{\mathrm{d}t}-2y=0, \quad x(0)=4, \; y(0)=0$

ANS $x=-\mathrm{e}^t(12t+13)+\mathrm{e}^{-3t}+16\mathrm{e}^{2t}, \quad y=-2\mathrm{e}^t(4t+3)-2\mathrm{e}^{-3t}+8\mathrm{e}^{2t}$

**7.21** $\dfrac{\mathrm{d}x}{\mathrm{d}t}-2x+y=5\mathrm{e}^t\cos t, \; x+\dfrac{\mathrm{d}y}{\mathrm{d}t}-2y=10\mathrm{e}^t\sin t, \; x(0)=y(0)=0$

ANS $x=5\mathrm{e}^t(1-\cos t+\sin t), \quad y=5\mathrm{e}^t(1-\cos t)$

**7.22** $\dfrac{dx}{dt} - 4x + 3y = \sin t, \quad 2x - \dfrac{dy}{dt} - y = 2\cos t, \quad x(0) = x_0,\ y(0) = y_0$

ANS $x = (-2x_0 + 3y_0 - 4)\,e^t + 3(x_0 - y_0 + 1)\,e^{2t} + \cos t - 2\sin t$

$y = (-2x_0 + 3y_0 - 4)\,e^t + 2(x_0 - y_0 + 1)\,e^{2t} + 2\cos t - 2\sin t$

**7.23** $\dfrac{dx}{dt} - 2x - y = 2e^t, \quad x - \dfrac{dy}{dt} + 2y = 3e^{4t}, \quad x(0) = x_0,\ y(0) = y_0$

ANS $x = \frac{1}{2}(x_0 + y_0 + 4)\,e^{3t} + \frac{1}{2}(x_0 - y_0 + 2t - 2)\,e^t - e^{4t}$

$y = \frac{1}{2}(x_0 + y_0 + 4)\,e^{3t} - \frac{1}{2}(x_0 - y_0 + 2t)\,e^t - 2e^{4t}$

**7.24** $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + \dfrac{dy}{dt} - 2y = 40e^{3t}, \quad \dfrac{dx}{dt} + x - \dfrac{dy}{dt} = 36e^t$

$x(0) = 1,\ y(0) = 3,\ x'(0) = 1$

ANS $x = 22e^{-2t} - 33e^{-t} - 3e^t(2t - 3) + 3e^{3t}, \quad y = 11e^{-2t} - 12e^t(t+1) + 4e^{3t}$

**7.25** $\dfrac{dx}{dt} - 2x - y = 2e^t, \quad \dfrac{dy}{dt} - 2y - 4z = 4e^{2t}, \quad x - \dfrac{dz}{dt} - z = 0$

$x(0) = 9,\ y(0) = 3,\ z(0) = 1$

ANS $x = 3t + 2 + 2e^t - 3e^{2t} + 8e^{3t}, \quad y = -6t - 1 - 4e^t + 8e^{3t}$

$z = 3t - 1 + e^t - e^{2t} + 2e^{3t}$

**7.26** $\dfrac{d^2x}{dt^2} + 2x - 2\dfrac{dy}{dt} = 0, \quad 3\dfrac{dx}{dt} + \dfrac{d^2y}{dt^2} - 8y = 240e^t$

$x(0) = y(0) = x'(0) = y'(0) = 0$

ANS $x = 12\cos 2t - 24\sin 2t - 10e^{-2t} + 30e^{2t} - 32e^t$

$y = -12\cos 2t - 6\sin 2t + 15e^{-2t} + 45e^{2t} - 48e^t$

**7.27** $\dfrac{dx}{dt} - x - 2y = 0, \quad x - \dfrac{dy}{dt} = 15\cos t\, H(t - \pi), \quad x(0) = x_0,\ y(0) = y_0$

ANS $x = \frac{2}{3}(x_0 + y_0)\,e^{2t} + \frac{1}{3}(x_0 - 2y_0)\,e^{-t} + \big[4e^{2(t-\pi)} + 5e^{-(t-\pi)}$

$+ 9\cos t + 3\sin t\big]H(t - \pi)$

$y = \frac{1}{3}(x_0 + y_0)\,e^{2t} - \frac{1}{3}(x_0 - 2y_0)\,e^{-t} + \big[2e^{2(t-\pi)} - 5e^{-(t-\pi)}$

$- 3\cos t - 6\sin t\big]H(t - \pi)$

**7.28** $\dfrac{dx}{dt} - x + y = 2\sin t\big[1 - H(t - \pi)\big], \quad 2x - \dfrac{dy}{dt} - y = 0, \quad x(0) = y(0) = 0$

ANS $x = (t+1)\sin t - t\cos t + \big[-(t - \pi + 1)\sin t + (t - \pi)\cos t\big]H(t - \pi)$

$y = 2(\sin t - t\cos t) + 2\big[-\sin t + (t - \pi)\cos t\big]H(t - \pi)$

**7.29** $2\dfrac{dx}{dt}+x-5\dfrac{dy}{dt}-4y=28e^t H(t-2), \quad 3\dfrac{dx}{dt}-2x-4\dfrac{dy}{dt}+y=0$

$x(0)=2, \; y(0)=0$

ANS $x=-e^{-t}+3e^t+\left[5e^{4-t}-(6t-7)e^t\right]H(t-2)$

$y=-e^{-t}+e^t+\left[5e^{4-t}-(2t+1)e^t\right]H(t-2)$

## *The Matrix Method*

Solve the following differential equations using the matrix method, in which $(\cdot)'=d(\cdot)/dt$.

**7.30** $x_1'=x_1-x_2, \quad x_2'=-4x_1+x_2$

ANS $x_1=C_1e^{-t}+C_2e^{3t}, \quad x_2=2C_1e^{-t}-2C_2e^{3t}$

**7.31** $x_1'=x_1-3x_2, \quad x_2'=3x_1+x_2$

ANS $x_1=e^t(A\cos 3t+B\sin 3t), \quad x_2=e^t(A\sin 3t-B\cos 3t)$

**7.32** $x_1'=5x_1+3x_2, \; x_2'=-3x_1-x_2, \; x_1(0)=1, \; x_2(0)=-2$

ANS $x_1=(-3t+1)e^{2t}, \quad x_2=(3t-2)e^{2t}$

**7.33** $x_1'=2x_1-x_2+x_3, \quad x_2'=x_1+2x_2-x_3, \quad x_3'=x_1-x_2+2x_3$

ANS $x_1=C_2e^{2t}+C_3e^{3t}, \quad x_2=C_1e^t+C_2e^{2t}, \quad x_3=C_1e^t+C_2e^{2t}+C_3e^{3t}$

**7.34** $x_1'=3x_1-x_2+x_3, \quad x_2'=x_1+x_2+x_3, \quad x_3'=4x_1-x_2+4x_3$

ANS $x_1=C_1e^t+C_2e^{2t}+C_3e^{5t}, \quad x_2=C_1e^t-2C_2e^{2t}+C_3e^{5t}$

$x_3=-C_1e^t-3C_2e^{2t}+3C_3e^{5t}$

**7.35** $x_1'=2x_1+x_2, \quad x_2'=x_1+3x_2-x_3, \quad x_3'=-x_1+2x_2+3x_3$

ANS $x_1=Ce^{2t}+e^{3t}(A\cos t+B\sin t), \; x_2=e^{3t}\left[(A+B)\cos t+(B-A)\sin t\right]$

$x_3=Ce^{2t}+e^{3t}\left[(2A-B)\cos t+(2B+A)\sin t\right]$

**7.36** $x_1'=3x_1-2x_2-x_3, \quad x_2'=3x_1-4x_2-3x_3, \quad x_3'=2x_1-4x_2$

ANS $x_1=(C_1+2C_2)e^{2t}+C_3e^{-5t}, \; x_2=C_2e^{2t}+3C_3e^{-5t}, \; x_3=C_1e^{2t}+2C_3e^{-5t}$

**7.37** $x_1'=x_1-x_2+x_3, \quad x_2'=x_1+x_2-x_3, \quad x_3'=-x_2+2x_3$

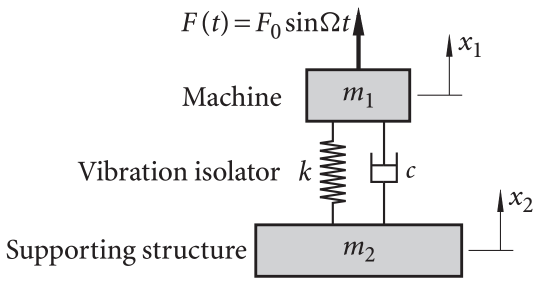
$x_1(0)=1, \quad x_2(0)=-2, \quad x_3(0)=0$

ANS $x_1=te^t+e^{2t}, \quad x_2=(t-2)e^t, \quad x_3=(t-1)e^t+e^{2t}$

**7.38** $x_1'=-x_1+x_2-2x_3, \quad x_2'=4x_1+x_2, \quad x_3'=2x_1+x_2-x_3$

ANS $x_1=\left[C_1+C_2(t-1)\right]e^{-t}, \quad x_2=-\left[2C_1+C_2(2t-1)\right]e^{-t}+2C_3e^t$

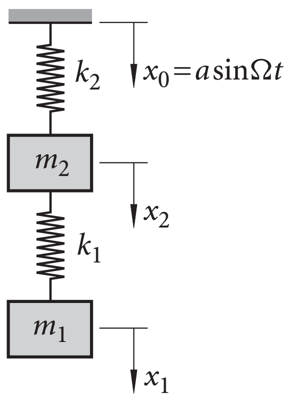
$x_3=-(C_1+C_2t)e^{-t}+C_3e^t$




2. For the case when $c \to 0$, show that the *steady-state* responses of both the machine and the supporting structure, i.e., $x_{1P}(t)$ and $x_{2P}(t)$ are given by

$$x_{1P}(t) = \frac{(k-m_2\Omega^2)F_0 \sin \Omega t}{m_1 m_2 \Omega^4 - k(m_1+m_2)\Omega^2}, \quad x_{2P}(t) = \frac{kF_0 \sin \Omega t}{m_1 m_2 \Omega^4 - k(m_1+m_2)\Omega^2}.$$

**8.5** A mass $m_1$ hangs by a spring of stiffness $k_1$ from another mass $m_2$ which in turn hangs by a spring of stiffness $k_2$ from the support as shown.




1. Show that the two natural circular frequencies of vibration are given by the equation

$$m_1 m_2 \omega^4 - \big[m_1(k_1+k_2) + m_2 k_1\big]\omega^2 + k_1 k_2 = 0.$$

2. If the support vibrates with $x_0(t) = a \sin \Omega t$, show that the amplitudes of the forced vibration are

$$a_1 = \frac{k_1 k_2 a}{m_1 m_2 \Omega^4 - \big[m_1(k_1+k_2) + m_2 k_1\big]\Omega^2 + k_1 k_2},$$

$$a_2 = \frac{(k_1 - m_1 \Omega^2) k_2 a}{m_1 m_2 \Omega^4 - \big[m_1(k_1+k_2) + m_2 k_1\big]\Omega^2 + k_1 k_2}.$$

**8.6** A mass $m$, supported by an elastic structure which may be modeled as a spring with stiffness $k$, is subjected to a simple harmonic disturbing force of maximum

### 12.1.3 The Laplace Transform

Integral transforms, such as the Laplace transform and the Fourier transform, are available by loading the `inttrans` package using `with(inttrans)`. In *Maple*, the Heaviside step function $H(t-a)$ is `Heaviside(t-a)`, and the Dirac delta function $\delta(t-a)$ is `Dirac(t-a)`.

`>with(inttrans):` ✍ Load the `inttrans` package.

✍ Given a function $f(t)$.

```
>f:=t*cosh(2*t)+t^2*sin(5*t)+t^3+sin(t)*Heaviside(t-Pi);
```

$$f := t\cosh 2t + t^2 \sin 5t + t^3 + \sin t \,\mathrm{Heaviside}(t-\pi)$$

`>F:=laplace(f,t,s);` ✍ Evaluate the Laplace transform using `laplace`.

$$F := \frac{1}{2(s-2)^2} + \frac{1}{2(s+2)^2} + \frac{10(3s^2-25)}{(s^2+25)^3} + \frac{6}{s^4} + \frac{e^{-\pi s}}{s^2+1}$$

✍ Given the Laplace transform of a function $G(s)$.

```
>G:=(s-3)/(s^2-6*s+25)+2/(s+2)^3+exp(-2*s)*(2+1/(s^2+1));
```

$$G := \frac{s-3}{s^2-6s+25} + \frac{2}{(s+2)^3} + e^{-2s}\left(2 + \frac{1}{s^2+1}\right)$$

✍ Evaluate inverse Laplace transform using `invlaplace`.

```
>g:=invlaplace(G,s,t);
```

$$g := e^{3t}\cos 4t + t^2 e^{-2t} + 2\,\mathrm{Dirac}(t-2) + \mathrm{Heaviside}(t-2)\sin(t-2)$$

✍ When evaluating the inverse Laplace transform by hand, one frequently needs to perform partial fraction decomposition, which can be easily done using *Maple*.

`>F:=8*(s+2)/(s-1)/(s+1)^2/(s^2+1)/(s^2+9);` ✍ Define a fraction.

$$F := \frac{8(s+2)}{(s-1)(s+1)^2(s^2+1)(s^2+9)}$$

✍ Perform partial fraction decomposition using `convert` with option `parfrac`, in which $s$ is the variable.

```
>convert(F,parfrac,s);
```

$$\frac{3}{10(s-1)} - \frac{1}{5(s+1)^2} - \frac{27}{50(s+1)} + \frac{s-3}{4(s^2+1)} - \frac{s-11}{100(s^2+9)}$$

When an ODE is solved using `dsolve` with the option `method=laplace`, *Maple* forces the equation to be solved by the method of Laplace transform.

## A.2 Table of Derivatives

1. $\dfrac{d}{dx}x^n = nx^{n-1}$

2. $\dfrac{d}{dx}e^x = e^x \implies \dfrac{d}{dx}a^x = a^x \ln a, \qquad \because\ a^x = e^{x\ln a}$

3. $\dfrac{d}{dx}\ln x = \dfrac{1}{x} \implies \dfrac{d}{dx}\log_a x = \dfrac{1}{x\ln a}, \quad \because\ \log_a x = \dfrac{\ln x}{\ln a}$

| Trigonometric Functions | Hyperbolic Functions |
|---|---|
| 4. $\dfrac{d}{dx}\sin x = \cos x$ | $\dfrac{d}{dx}\sinh x = \cosh x$ |
| 5. $\dfrac{d}{dx}\cos x = -\sin x$ | $\dfrac{d}{dx}\cosh x = \sinh x$ |
| 6. $\dfrac{d}{dx}\tan x = \dfrac{1}{\cos^2 x} = \sec^2 x$ | $\dfrac{d}{dx}\tanh x = \dfrac{1}{\cosh^2 x} = \text{sech}^2 x$ |
| 7. $\dfrac{d}{dx}\cot x = -\dfrac{1}{\sin^2 x} = -\csc^2 x$ | $\dfrac{d}{dx}\coth x = -\dfrac{1}{\sinh^2 x} = -\text{csch}^2 x$ |
| 8. $\dfrac{d}{dx}\sec x = \dfrac{\sin x}{\cos^2 x} = \tan x \sec x$ | $\dfrac{d}{dx}\text{sech}\, x = -\dfrac{\sinh x}{\cosh^2 x} = -\tanh x\, \text{sech}\, x$ |
| 9. $\dfrac{d}{dx}\csc x = -\dfrac{\cos x}{\sin^2 x} = -\cot x \csc x$ | $\dfrac{d}{dx}\text{csch}\, x = -\dfrac{\cosh x}{\sinh^2 x} = -\coth x\, \text{csch}\, x$ |

**Inverse Trigonometric Functions**

| | |
|---|---|
| 10. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$ | $\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$ |
| 11. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$ | $\dfrac{d}{dx}\cot^{-1} x = -\dfrac{1}{1+x^2}$ |
| 12. $\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$ | $\dfrac{d}{dx}\csc^{-1} x = -\dfrac{1}{x\sqrt{x^2-1}}$ |