University of Waterloo

Department of Civil Engineering

CIV E 205 – Mechanics of Solids II

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Course Notes

University of Waterloo

Civil Engineering

CIV. E. 205 – MECHANICS OF SOLIDS II

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Lectures: MWF 9:30 - CPH 3385, Turotials; Wednesdays 2:30-5:30

Course Site: http://www.civil.uwaterloo.ca/tarek/205-2007.html

Course Material: Booklet of Hibbeler, 2005 "**Mechanics of Materials**," 6th Edition, Prentice Hall (Book store), and Course Notes (download)

Tentative Course Material:

1. Internal loadings on beams and frames
2. Stresses on beams and frames
3. Stress/strain transformation
4. Mohr's circle for stress and strain
5. Strain Rosettes
6. Generalized Hooke's law
7. Theories of failure
8. Deflection using integration method
9. Moment - Area Method
10. Strain Energy
11. Virtual Work
12. Statically indeterminate beams and frames
13. Castigliano's Theorem
14. Buckling
15. Influence Lines

Marking:

Tutorial Exercises:	10%	Checked at the end of tutorials
4 Quizzes @ 10%:	40%	Held on dates announced in class
Final Examination:	50%	
Bridge Competition:	Bonus	

Notes:

- Each week, a number of suggested problems will be given to serve as background study for the quizzes. Solutions are **not** to be handed in.

- Teaching Assistants will provide one-to-one help and will prepare you for quizzes.

- Course notes, solutions to suggested problems, and solutions to quizzes will be posted on the course web site.

Table of Contents

		Page
	Basic Concepts	4
1.	Internal loadings on beams and frames	7
2.	Stresses on beams and frames	13
3.	Stress/strain transformation	17
4.	Mohr's circle for stress and strain	19
5.	Strain Rosettes	26
6.	Generalized Hooke's law	28
7.	Theories of failure	31
8.	Deflection using integration method	34
9.	Moment – Area Method	37
10.	Strain Energy Method	44
11.	Virtual Work Method	47
12.	Statically indeterminate beams and frames	50
13.	Castigliano's Theorem	51
14.	Buckling	52
15.	Influence Lines	

Mechanics of Materials

Objectives:

- Solve Problems in a structured systematic manner;
- Study the behavior of bodies that are considered deformable under different loading conditions; &
- Analyze and design various machines / systems

Basic Concepts

a) Equilibrium of a system subjected to Forces

(i.e., Resultant of all forces on the system = 0)

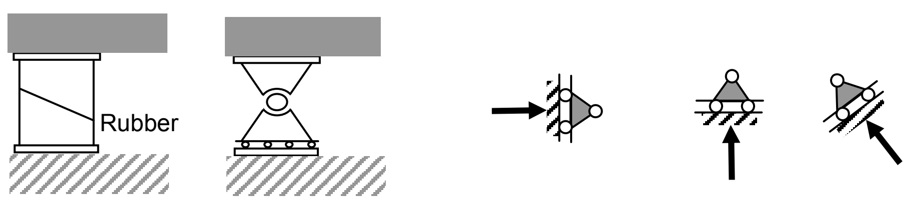
Three Equilibrium Conditions:

1. $\overset{+}{\rightarrow}\sum X$ components of all forces = 0
2. $\overset{+}{\uparrow}\sum Y$ components of all forces = 0
3. $\overset{+}{\curvearrowright}\sum M$ (moment at any point) = 0

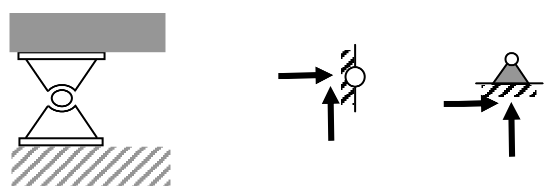
b) Types of Supports

Supports exert reactions in the direction in which they restrain movement.

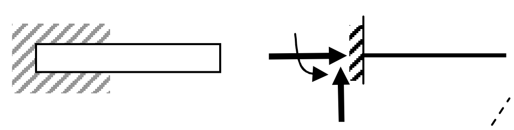
Roller Support (restricts in one direction only and allows rotation)



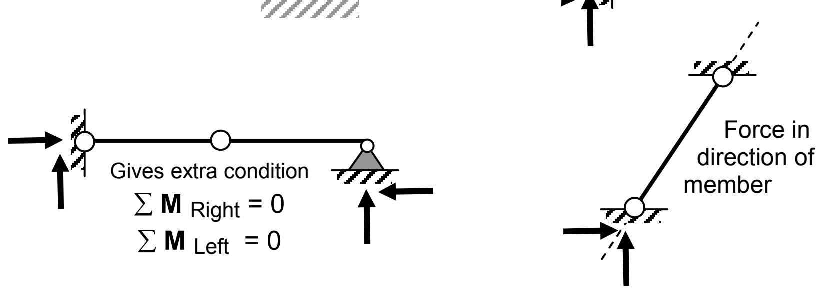
Hinged or Pinned Support (restricts in two ways and allows rotation)



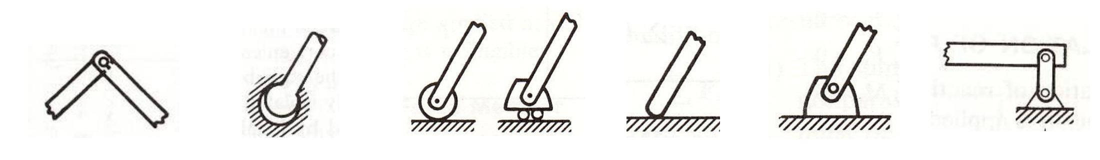
Fixed Support (restricts in two directions and also restricts rotation)



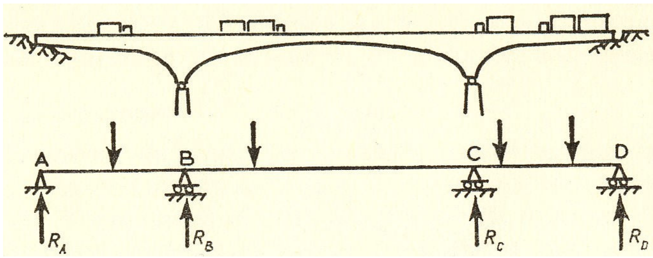
Intermediate Pin or Hinge (Gives one extra condition)

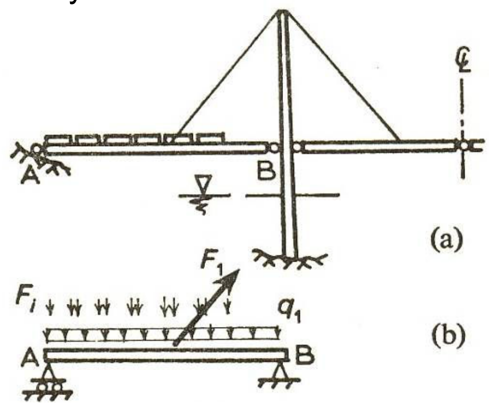


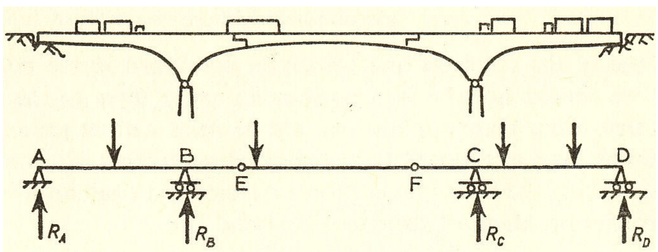
Examples:

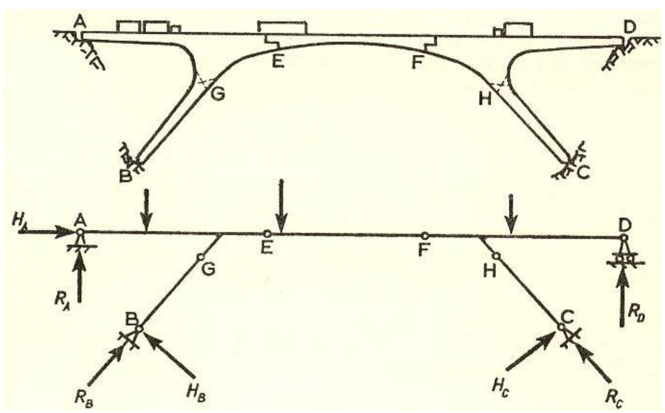


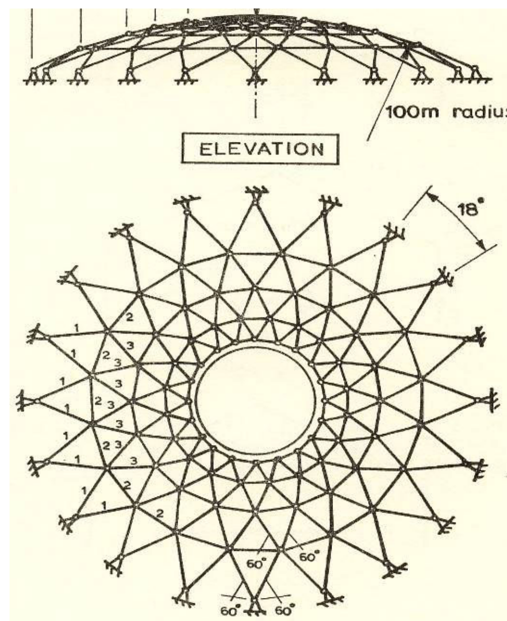
c) Structural Representation of Real Systems





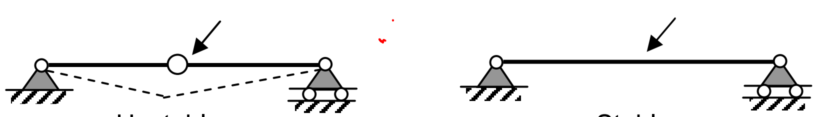






d) Stability & Determinacy of Structures

- A stable structure can resist a general force immediately at the moment of applying the force.



- A statically determinate structure is when the reactions can be determined using equilibrium equations.

1. Beams:

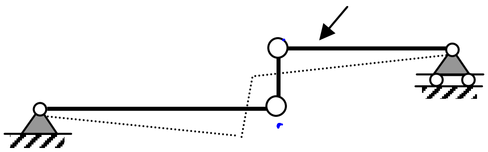
r = unknown support **r**eactions.

c = additional **c**onditions

if r < c + 3 Unstable

if r = c + 3 Statically determinate

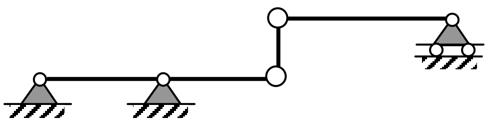
if r > c + 3 Statically Indeterminate



r = 3 (two at hinge + one at roller)

c = 2 (two intermediate hinges), then,

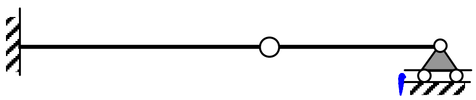
r < c + 3 Unstable



r = 5 (four at hinges + one at roller)

c = 2 (two intermediate hinges), then,

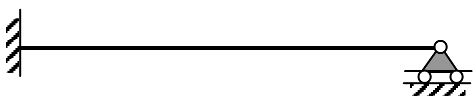
r = c + 3 Stable & Statically Determinate



r = ________

c = ________

r = ________, then____________________



r = 4 (three at fixed end + one at roller)

c = 0, then

r > c + 3 Stable & Statically Indeterminate

2. Frames:

j = No. of **j**oints

m = No. of **m**embrs

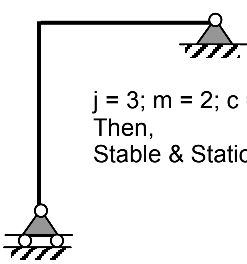
r = unknown support **r**eactions

c = special **c**onditions

if 3m + r < 3j + c Unstable

if 3m + r = 3j + c Statically determinate

if 3m + r > 3j + c Statically Indeterminate



j = 3; m = 2; c = 0; r = 3

Then,

Stable & Statically determinate

3. Trusses:

j = No. of **j**oints

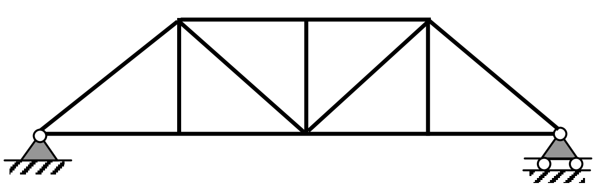
m = No. of **m**embrs

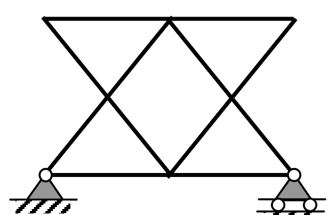
r = unknown support **r**eactions

if m + r < 2j Unstable

if m + r = 2j Statically determinate

if m + r > 2j Statically Indeterminate





j = 8; m = 12; r = 3

Then,

m + 3 = 15 < 2j, or Unstable

1. Internal Loadings on Beams & Frames

Step 1: Get Support Reactions

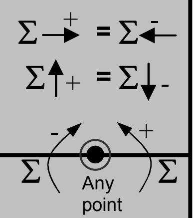
(Load on the **Whole** structure is carried by the supports)

Sign Convention for Equilibrium:

$\sum F_x = 0$

$\sum F_y = 0$

$\sum M = 0$



Considering the **whole** structure

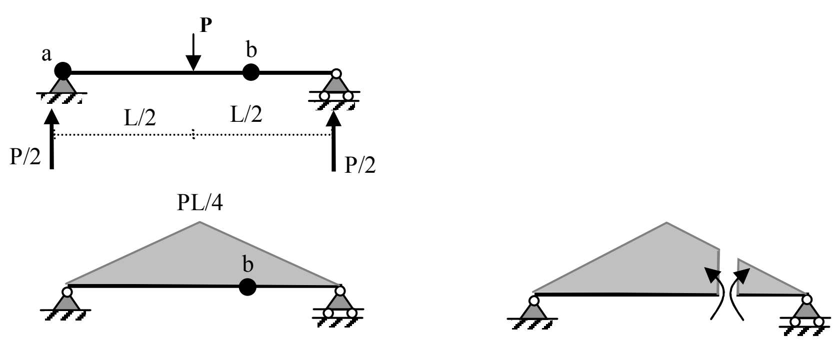
Step 2: Get Internal Forces at various points

(Load to the **left side** of point = Loads to the **write side** of point)

Sign Convention for Internal Forces:

B.M.D.
S.F.D.
N.F.D

Important Note:



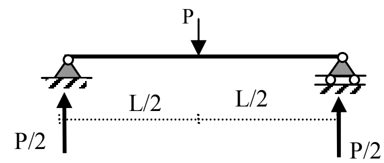
A section at point **b** shows that the internal bending moment (from each side separately) has a positive sign. Yet, it is in equilibrium from both sides.

Important Rule: To get the internal forces (B.M. & S.F.) we always calculate from one side.

Internal Forces: B.M.D., S.F.D., & N.F.D.

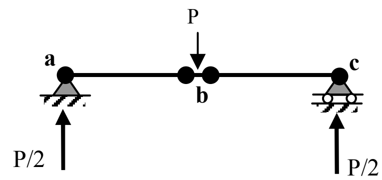
Reactions:

1. Check stability of the structure.
2. Assume directions for the reactions and apply **Equilibrium** equations at any points, considering the whole structure (i.e., **both sides** around any point).
3. Get reactions with correct directions. Check the equilibrium of a new point to make sure reactions are OK.

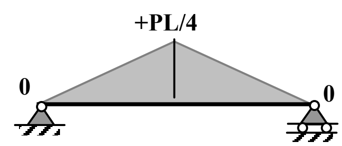


B.M.D.:

4. Identify points of change in load or shape.
5. Calculate the moment at each point, considering only **one side** of the structure and the sign convention.
   i.e., Left of point **a**, B.M. = 0;
   Right of point **c**, B.M. = 0; and
   Either left or right of point **b**, B.M. = + P. L / 4
6. Draw the B.M.D. using the values calculated in step 5, then connect these values.
7. Check if the B.M.D. is logical.



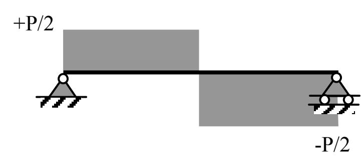
B.M.D.



S.F.D. and N.F.D.:

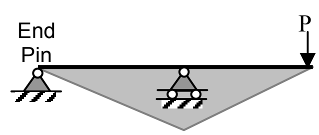
8. Start from the left of the structure and draw the total values to the left of each point, following the load changes and the sign convention.

S.F.D.

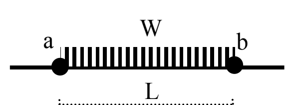


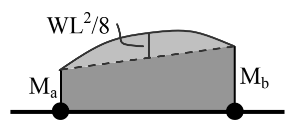
Rules:

1. B.M. at free end = 0



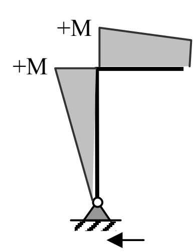
2. Any support has B.M. on top of it, unless it is an end pin, end roller, or an intermediate pin.
3. The B.M. at the **middle** of a UDL is $+wL^2/8$.

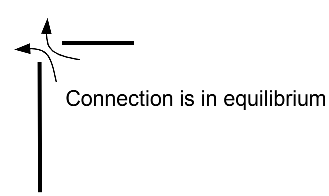




Special case: simple beam

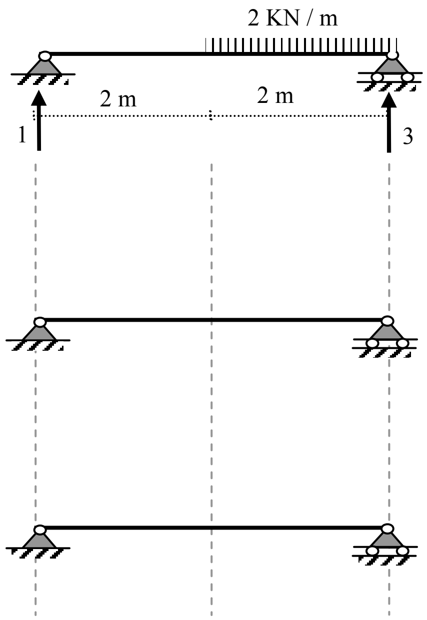
4. Any connection has same B.M. (value and sign) at its two sides.
5. Shear curve is one higher degree than load curve.
6. Moment curve is one higher degree than shear curve.
7. Moment is maximum at the point where shear = 0.



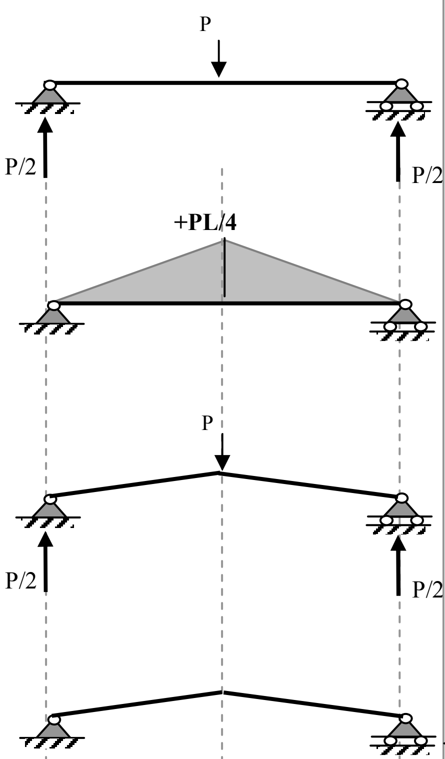


8. Between any two points:
   - Area under load = difference in shear
   - Area under shear = difference in moment
   - Slope of shear curve = - (load trend)
   - Slope of moment curve = shear trend

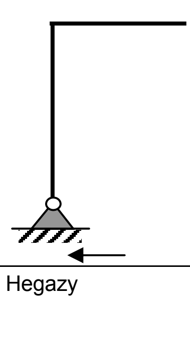
Examples:

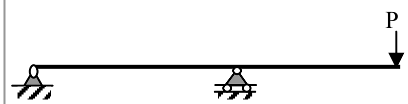


2 KN / m
2 m
2 m
1
3

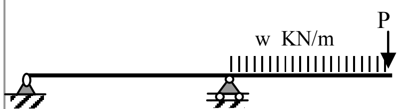


P
P/2
P/2
+PL/4
P
P/2
P/2

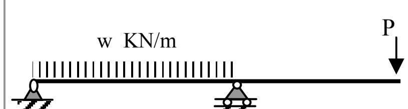




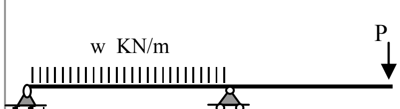
P



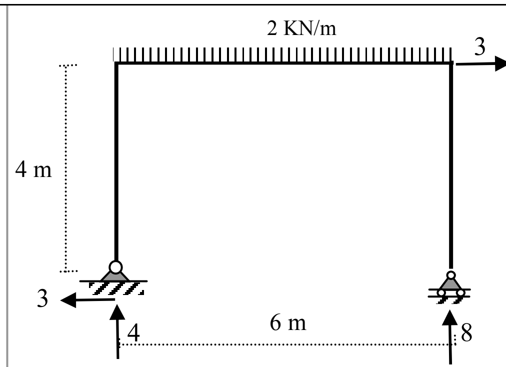
w KN/m
P



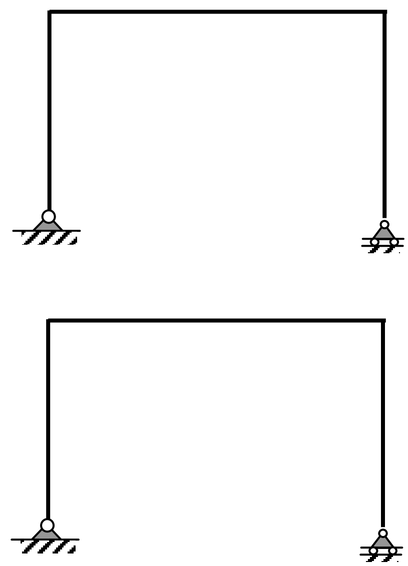
w KN/m
P

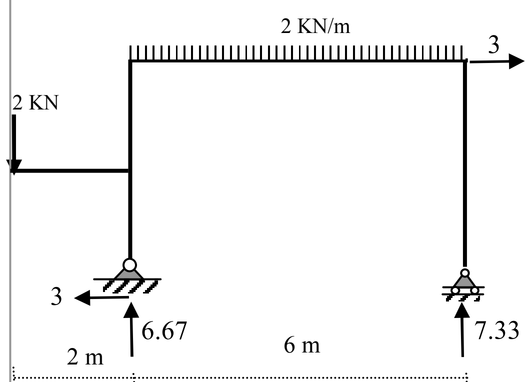


w KN/m
P

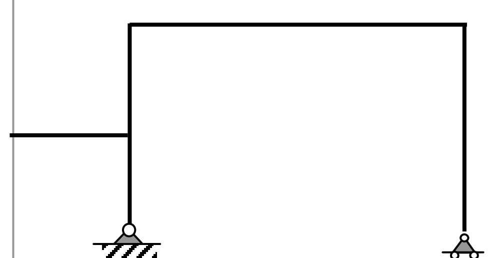


2 KN/m
3
4 m
3
4
6 m
8

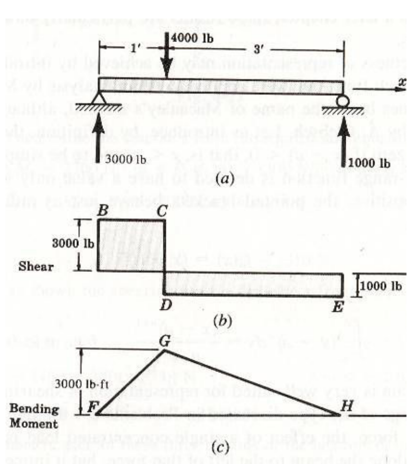


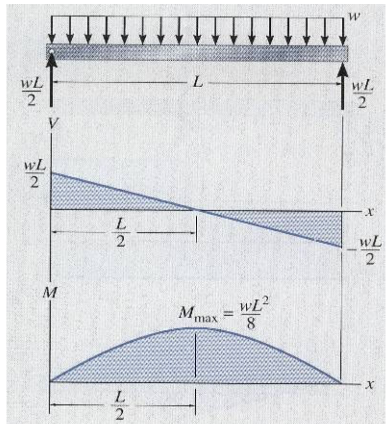


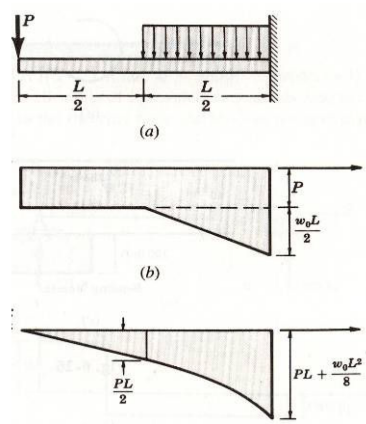
2 KN/m
3
2 KN
3
6.67
6 m
7.33
2 m

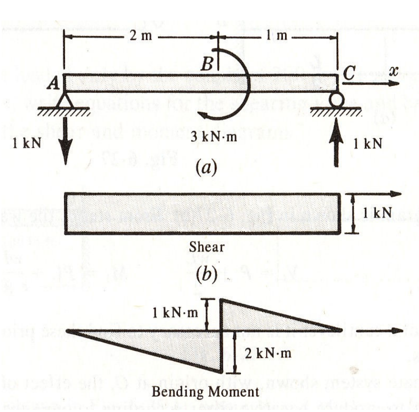


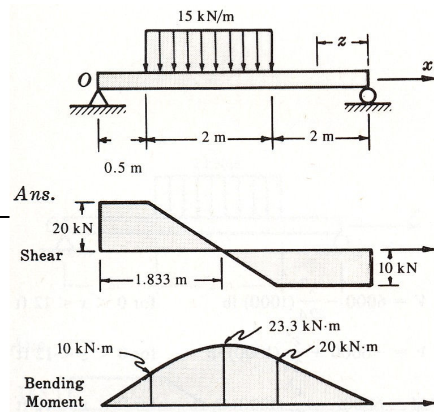
Examples: Calculate and draw the S.F.D. and the B.M.D.

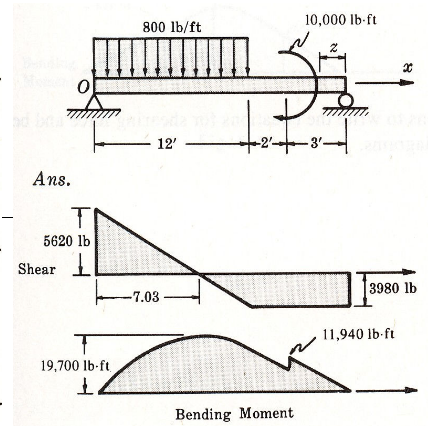


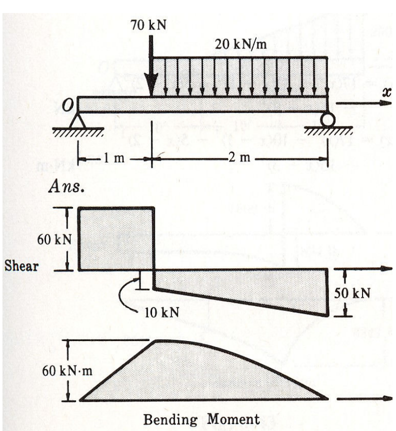


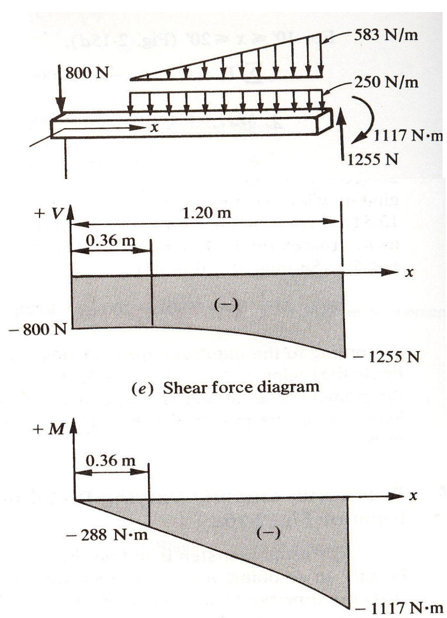


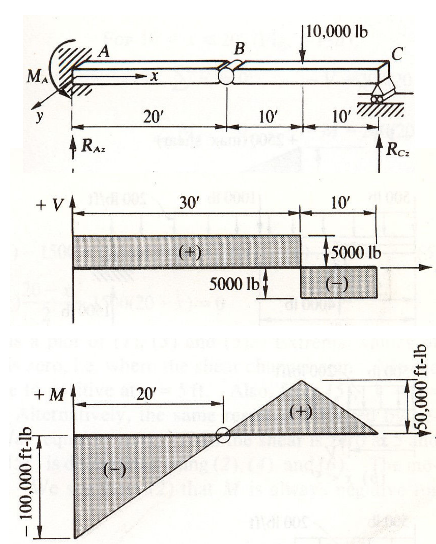










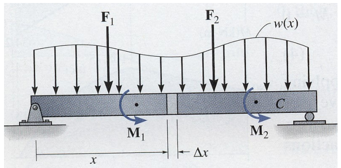


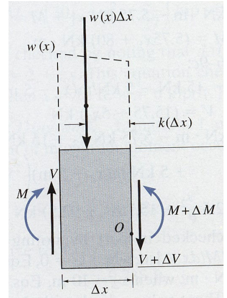
Solved examples 6-1 to 6-6

Note on simple beam with distributed load:

Graphical Approach:

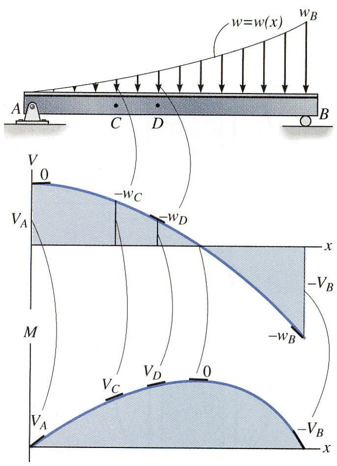
Stability & Determinacy - Reactions – N, V, & M Relations – Draw Diagrams





Free-body diagram of segment Δx
(b)

Examples on Page 10

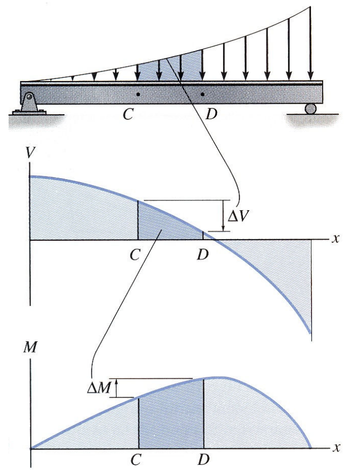


$$\frac{dV}{dx} = -w(x)$$

slope of shear diagram at each point = −distributed load intensity at each point

$$\frac{dM}{dx} = V$$

slope of moment diagram at each point = shear at each point

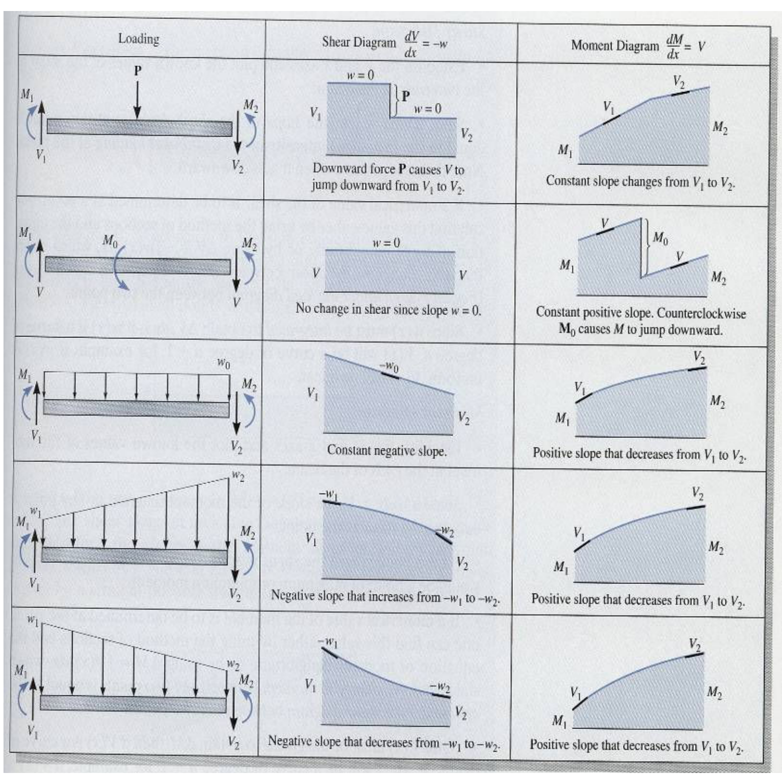


$$\Delta V = -\int w(x)\, dx$$

change in shear = −area under distributed loading

$$\Delta M = \int V(x)\, dx$$

change in moment = area under shear diagram



Rules:

1- Shear curve is one degree above load curve

2- Moment curve is one degree above shear curve

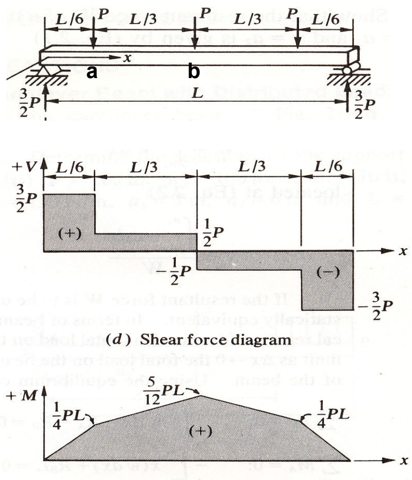
3- Moment is maximum at point with shear = 0

4- Between any two points: (look at table)

- Area under load = difference in shear
- Area under shear = difference in moment
- Slope of shear curve = - (load trend)
- Slope of moment curve = shear trend

Solved examples 6-7 to 6-13

Examples: Calculate and draw the S.F.D. and the B.M.D.



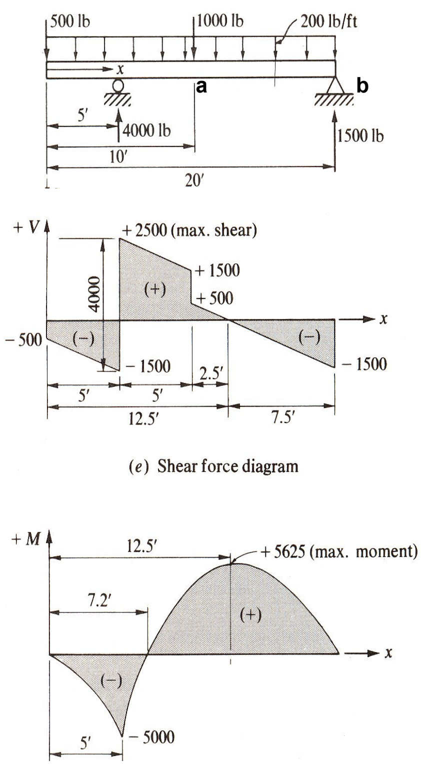
(d) Shear force diagram

(e) Bending moment diagram

From point **a** to point **b**:

- Load curve =
- Shear curve =
- Moment curve =
- Area under load = = difference in shear =
- Area of shear =

  = difference in moment =
- Shear at point of max. Moment =
- Max. moment can be calculated from shear diagram = =
- Slope of shear curve =
- Slope of moment curve =



(e) Shear force diagram

From point **a** to point **b**:

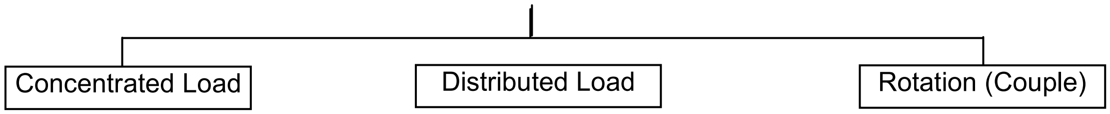
- Load curve =
- Shear curve =
- Moment curve =
- Area under load =

  = difference in shear =
- Area of shear =

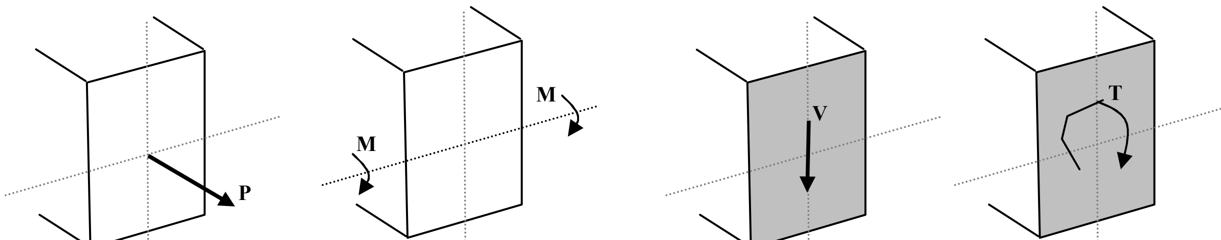
  = difference in moment =
- Shear at point of max. Moment =
- Max. moment can be calculated from shear diagram = =
- Slope of shear curve =
- Slope of moment curve =

2. Stresses on Beams and Frames

Forces and their effects at different points:



Types of Forces on a Cross-Section:



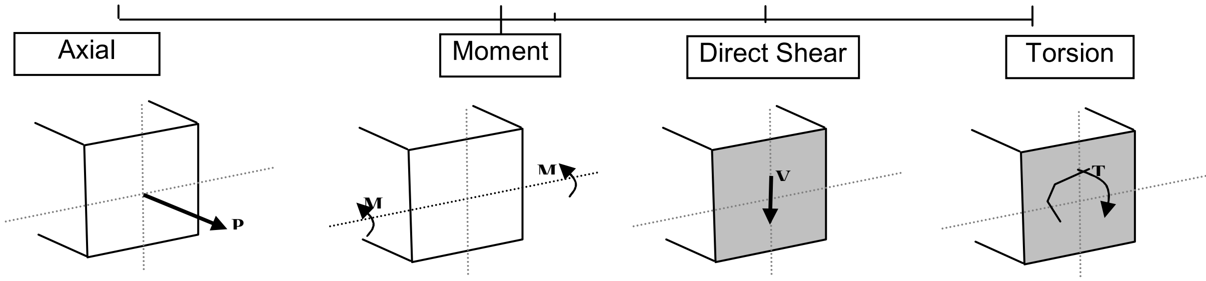
Axial Force = Perpendicular to X-Section

Bending Moment = Couple Normal to Plane

Direct Shear = Force in the X-Section Plane

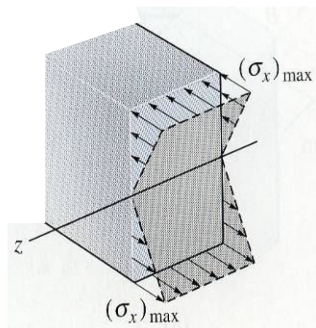
Torsion = Couple in the X-Section Plane

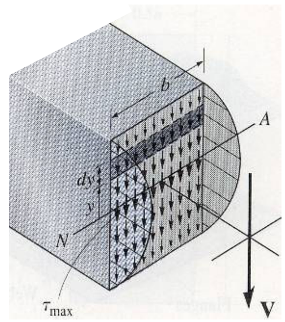
Types of Forces:

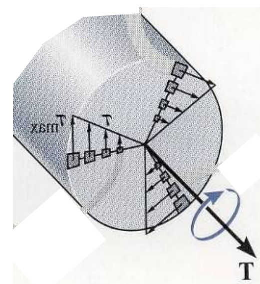


Normal Stresses

Shear Stresses



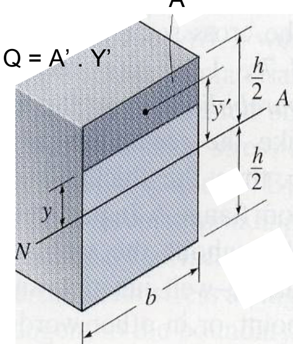




Stresses $\sigma = \frac{P}{A}$ $\sigma = \frac{-My}{I}$ $\tau = \frac{VQ}{It}$ $\tau = \frac{T\rho}{J}$

A’

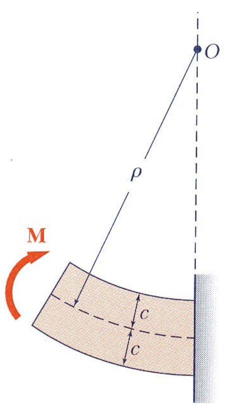
Q = A’ . Y’



$J = \frac{\pi}{2}c^4$

In narrow rectangular beams,

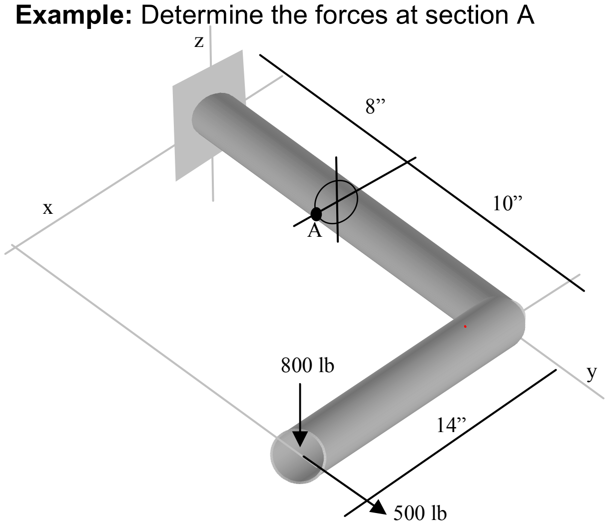
τ_{max} = 1.5 V / A



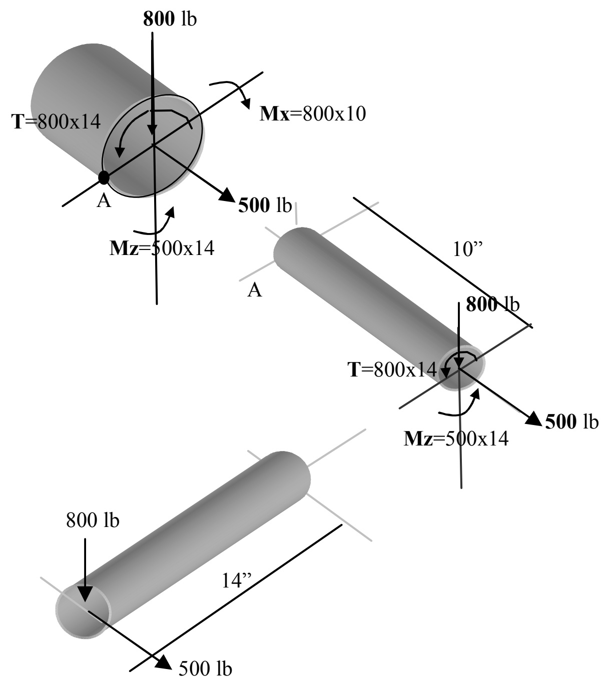
$\frac{1}{\rho} = \frac{M}{EI}$

Shape of cross section	τ_{max}	ϕ
Square	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4 G}$
Equilateral triangle	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4 G}$
Ellipse	$\frac{2\,T}{\pi a b^2}$	$\frac{(a^2+b^2)TL}{\pi a^3 b^3 G}$

Example: Determine the forces at section A



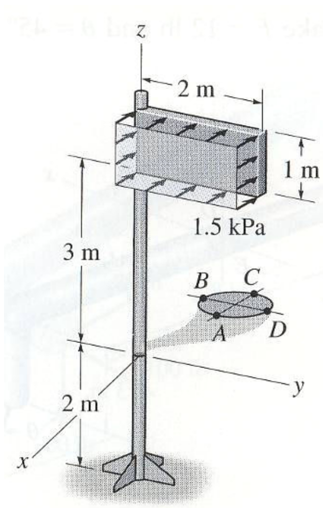
Note: When the structural system is:__________, then the **free end** is a good starting point for the analysis.



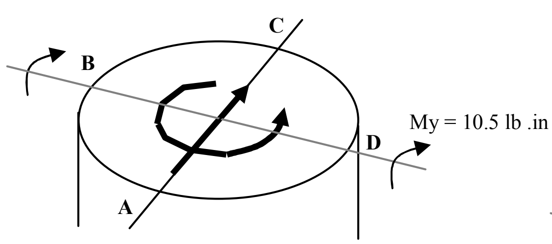
Equilibrium equations for each segment:

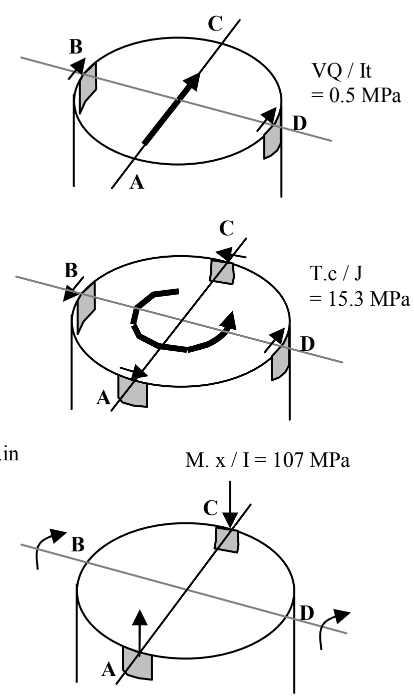
∑Mx=0, ∑My=0, ∑Mz=0
∑Fx=0, ∑Fy=0, ∑Fz=0

Example: Determine the internal stresses at points A, B, C, & D



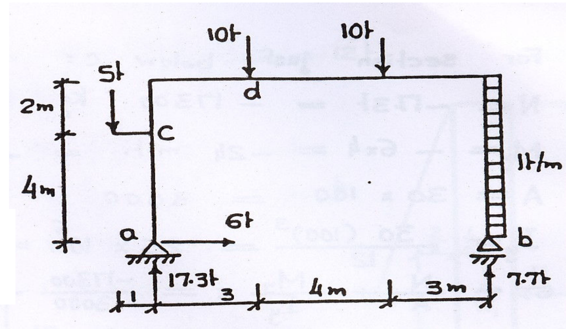
V = 3 KN
T = 3 KN
My = 10.5 KN.m





Example:

Calculate normal stresses at section d and also at the section just below c.



First, we get the reactions.

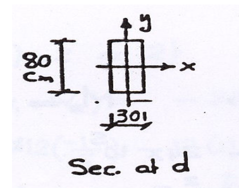
For section d: $N =$

$M_x =$

$A = 30 \times 80 = 2400 \ cm^2$

$I_x = \frac{30\,(80)^3}{12} = 128 \times 10^4 \ cm^4$

$\sigma = \frac{N}{A} - \frac{M_x}{I_x}\, y = \frac{-6000}{2400} - \frac{-4.1 \times 10^5}{128 \times 10^4}\, y$



$\sigma_{top} = -2.5 + 0.32 \times 40 = +10.3 \ kg/cm^2$

$\sigma_{bot.} = -2.5 + 0.32\,(-40) = -15.3 \ kg/cm^2$

For section just below c:

$N =$

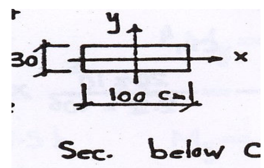
$M_y =$

$A = 30 \times 100 = 3000 \ cm^2$

$I_y = \frac{30\,(100)^3}{12} = 2.5 \times 10^6 \ cm^4$

$\sigma = \frac{N}{A} - \frac{M_y}{I_y}\, x = \frac{-17300}{3000} - \frac{24 \times 10^5}{2.5 \times 10^6}\, x$

$= -5.8 - 0.96\, x$



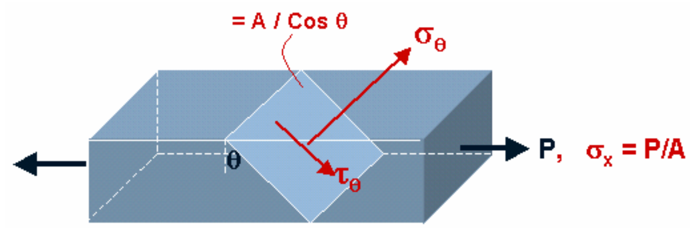
$\sigma_{right} = -5.8 - 0.96 \times 50 = -53.8 \ kg/cm^2$

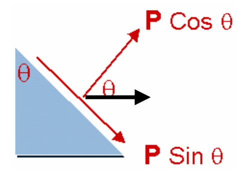
$\sigma_{left} = -5.8 - 0.96\,(-50) = +42.2 \ kg/cm^2$

Solved Problems 6-14 to 6-20, 7-1 to 7-3, 8-4 to 8-6

3. Transformation of Stresses

- Member under **tension only (P) in one direction**, i.e., a normal stress. But, let's consider an inclined plane.





σ_θ = (P Cos θ) / (A / Cos θ) or

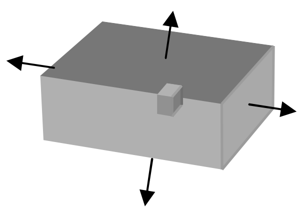
$\sigma_\theta = \sigma_x \cos^2\theta$

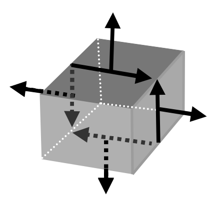
$\tau_\theta = ½\ \sigma_x \sin 2\theta$

Very important conclusions:

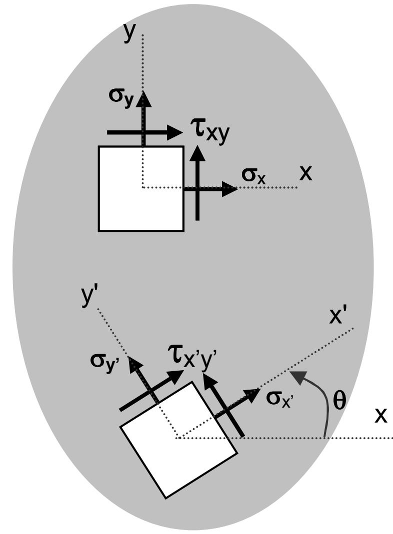
- Under tension only, **shear** is automatically present at various planes.
- The plane of maximum shear is when Sin 2θ = max or when θ = 45.
- Maximum shear = σ_x /2 = P / 2A
- It is important to study stress transformation and shear failure.

- Member under two dimensional stresses.





Positive Signes



Questions:
Is this the maximum stress? **If not**, then
What is the value of max. normal stress & its orientation? and
What is the value of maximum shear stress & its orientation?

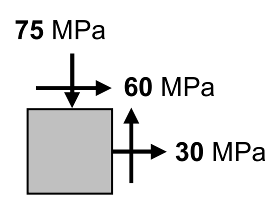
General Equations:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

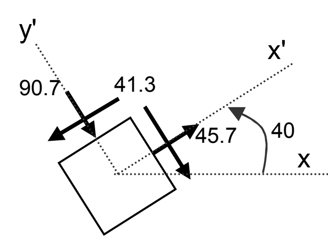
Example:
For the given state of stress, determine the normal and shearing stresses after an element has been rotated 40 degrees counter-clockwise.



σ_x = +30 MPa ; σ_y = -75 MPa ; τ_{xy} = +60 MPa ; θ = + 40

Applying the above equations, we get:

$\sigma_{x'}$ = +45.7 MPa ; $\sigma_{y'}$ = -90.7 MPa ; $\tau_{x'y'}$ = -41.3 MPa



Solved Examples 9-2 to 9-6

Important Observations:

1. $\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$ = Constant
   Sum of normal stress is constant (90 degrees apart) for any orientation.

2. The plane in which shear stress $\tau_{x'y'}$ = 0 is when:

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = 0$$

   or tan 2θ = 2 τ_{xy} / ($\sigma_x - \sigma_y$) or at θ_1, θ_2 having 90 degrees apart. These are called **principal planes**.

3. $\sigma_{x'}$ becomes maximum when $d\sigma_{x'} / d\theta = 0$, or when differentiating the following equation:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

   we get, $\tan 2\theta p = 2 \tau_{xy} / (\sigma_x - \sigma_y)$ or, exactly at the principal planes, which has shear stress = 0.
   The value of the principal normal stresses are:

$$\sigma_{max, min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

4. Since $\sigma_x + \sigma_y$ = constant, then, at the principal planes, σ_x is maximum but σ_y is minimum.

5. $\tau_{x'y'}$ is maximum when planes, $d\tau / d\theta = 0$, or when:

   $\tan 2\theta s = -(\sigma_x - \sigma_y) / 2 \tau_{xy}$ and the value of maximum shear stress τ_{xy} is:

$$\tau_{x'y'} \max = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

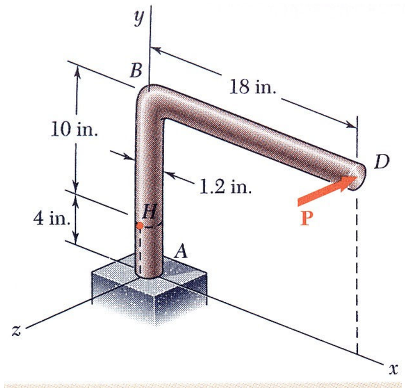
6. Similar to single stress situation, maximum is when $d\tau / d\theta = 0$, or when: θ = _____.

Example:

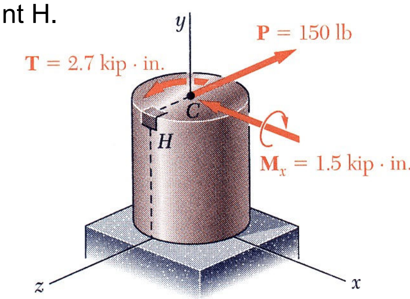
Check rule 1 for the example in previous page.

In the general equations, even if the original τ_{xy} on the element = 0, then still the shear at any plane ($\tau_{x'y'}$)has a value as a function of normal stresses.

Example: Determine the maximum normal and shear stresses at point H.



Forces at the section:

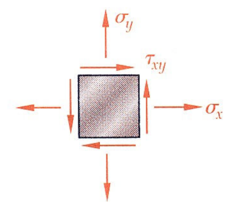


Stresses at Point H:

$$\sigma_y = +\frac{Mc}{I} = +\frac{(1.5 \text{ kip} \cdot \text{in.})(0.6 \text{ in.})}{\frac{1}{4}\pi\,(0.6 \text{ in.})^4} \qquad \sigma_y = +8.84 \text{ ksi}$$

$$\tau_{xy} = +\frac{Tc}{J} = +\frac{(2.7 \text{ kip} \cdot \text{in.})(0.6 \text{ in.})}{\frac{1}{2}\pi\,(0.6 \text{ in.})^4} \qquad \tau_{xy} = +7.96 \text{ ksi}$$

$$\sigma_x = 0$$



Principal stresses:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(7.96)}{0 - 8.84} = -1.80$$

$$2\theta_p = -61.0^\circ \quad \text{and} \quad 180^\circ - 61.0^\circ$$

$$\theta_p = -30.5^\circ \quad \text{and} \quad +59.5^\circ$$

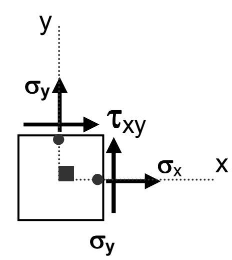
$$\sigma_{\text{max, min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{0 + 8.84}{2} \pm \sqrt{\left(\frac{0 - 8.84}{2}\right)^2 + (7.96)^2} = +4.42 \pm 9.10$$

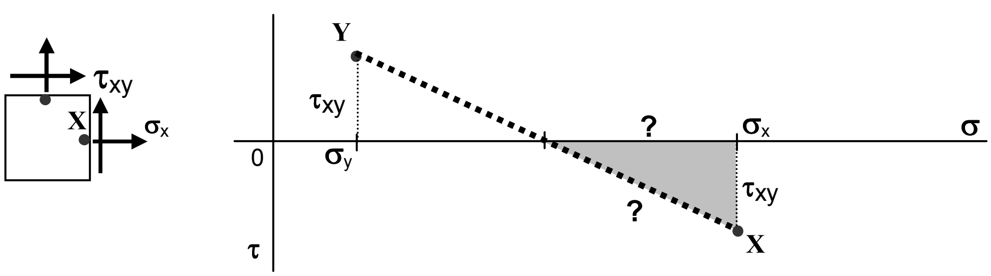
$$\sigma_{\text{max}} = +13.52 \text{ ksi} \qquad \sigma_{\text{min}} = -4.68 \text{ ksi}$$

4. Circular representation of plane stresses (Mohr's Circle):

Given a state of stress, with σ_x and σ_y having 90 degrees apart.

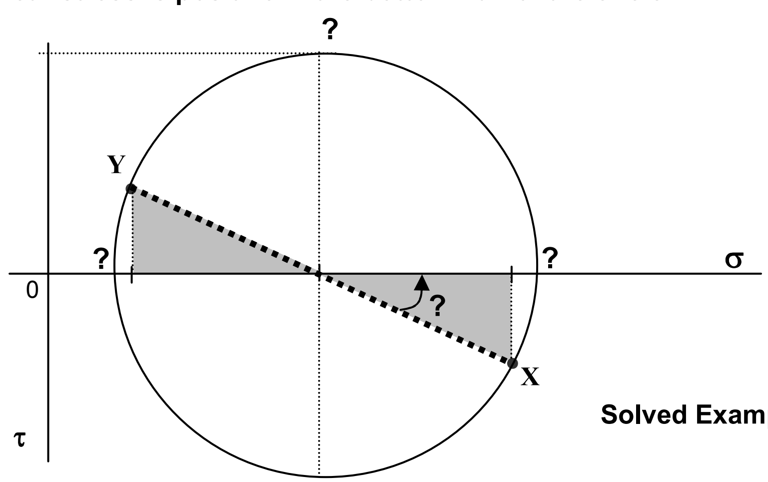


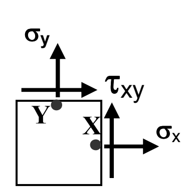
Step 1: Let's **plot** the two points X and Y.



Step 2: Draw a circle from the center to pass by points **X** and **Y**. Determine σ_{max}, σ_{min}, θ_p, τ_{max}, θ_s

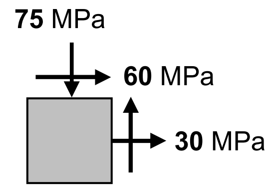
Notice that Shear stress is positive in the bottom half of the circle.

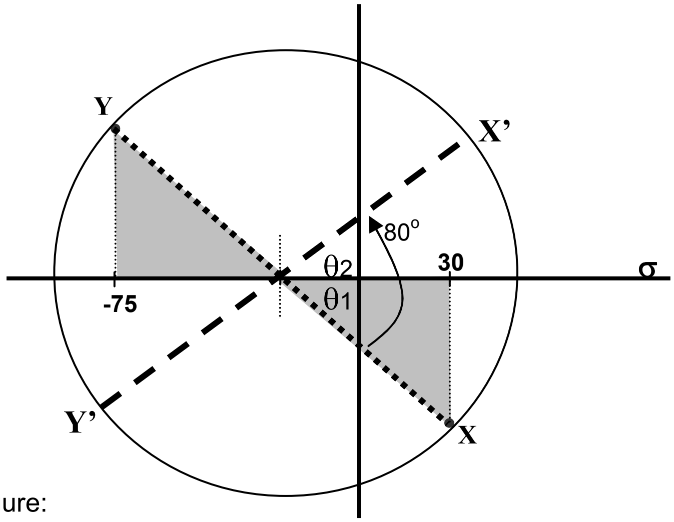


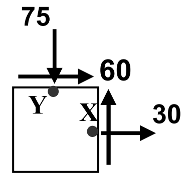


Solved Examples 9-7 to 9-13

Example:
For the given state of stress, determine the normal and shearing stresses after an element has been rotated 40 degrees counter-clockwise.







From the figure:

Average stress = Center of circle = $(30 - 75)/2 = -22.5$, R = sqrt $(52.5^2 + 60^2) = 79.7$
$\tan \theta 1 = 60 / 52.5$, then $\theta 1 = 48.8^o$ and $\theta 2 = 80 - \theta 1 = 31.2^o$

Then, points X' and Y' have the following coordinates:

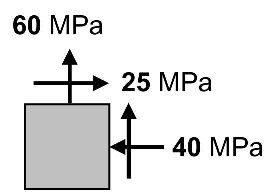
$\sigma_{x'} = -22.5 + R \cos \theta 2 = -22.5 + 79.9 * 0.855 = +45.7$ MPa

$\sigma_{y'} = -22.5 - R \cos \theta 2 = -90.7$ MPa ; $\tau_{x'y'} = R \sin \theta 2 = -41.3$ MPa

Principal stress values:

$\sigma_{max}, \sigma_{min}$ = Average ± R = -22.5 ± 79.7 = 57.2, - 102.2

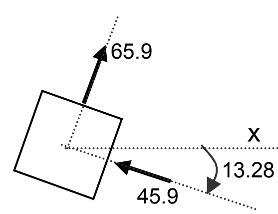
Example:
For the given state of stress, determine: a) principal planes; and b) principal stresses.



Analytically: $\sigma_x = -40$ MPa; $\sigma_y = +60$ MPa ; $\tau_{xy} = +25$ MPa

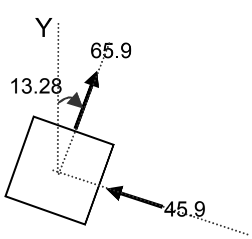
$\tan 2\theta_p = 2\, \tau_{xy} / (\sigma_x - \sigma_y) = 2 \times 25 / (-40 - 60) = -0.5$
or at $\theta_{p1} = -13.28$; $\theta_{p2} = 76.7$

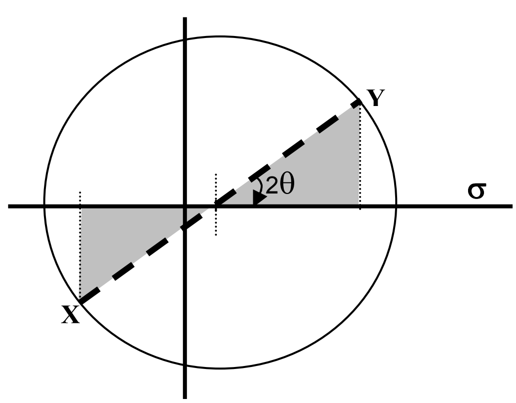
$$\sigma_{max}, \sigma_{min} = \text{Average} \pm R = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}} = 10 \pm 55.9 \text{ MPa}$$

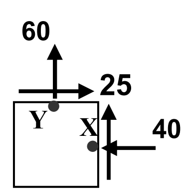


Graphically: Two points X & Y

Center =
R =
$\sigma_{max}, \sigma_{min}$ = Average ± R

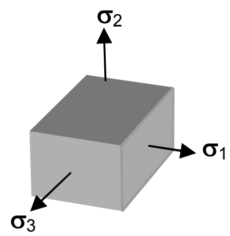


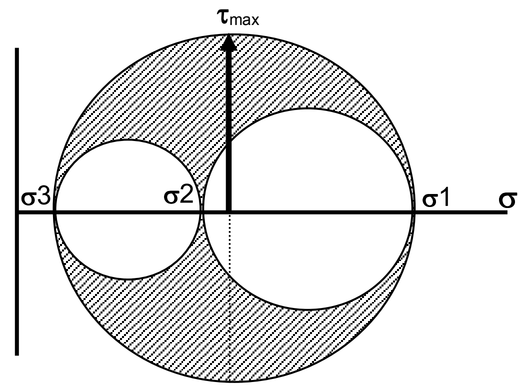




3-Dimensional stress systems: (Absolute maximum shear stress)

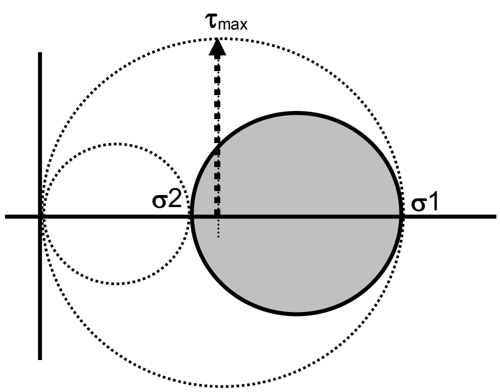
Assume $\sigma 1 > \sigma 2 > \sigma 3$ are principal normal stresses (no shear), then let's draw Mohr's circle.



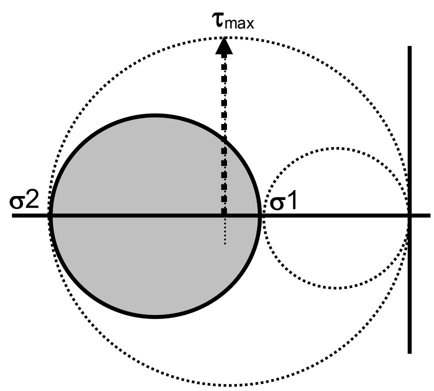


Note : Even if $\sigma 3 = 0$, 3-D stress analysis becomes essential.

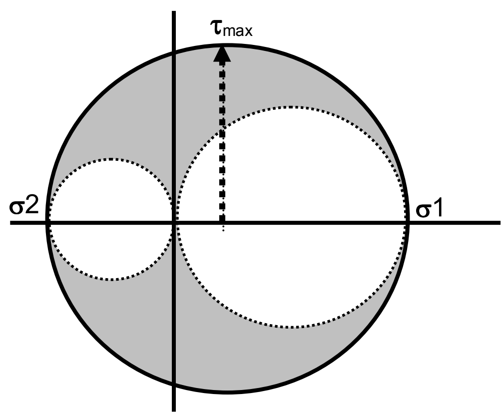
Case 1: both $\sigma 1$ and $\sigma 2$ are positive
Then, $\tau_{max} = \sigma 1 / 2$



Case 2: both $\sigma 1$ and $\sigma 2$ are negative
Then, $\tau_{max} = \sigma 2 / 2$



Case 3: $\sigma 1$ and $\sigma 2$ have opposite signs
Then, $\tau_{max} = (\sigma 1 - \sigma 1) / 2$



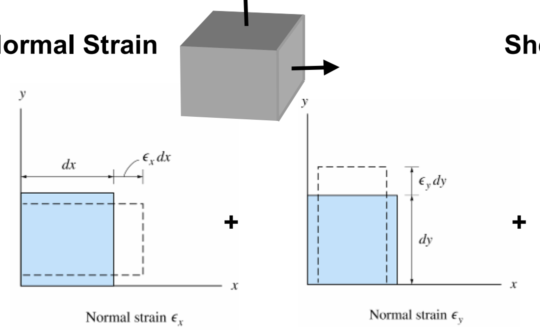
Examples

Transformation of Plain Strain

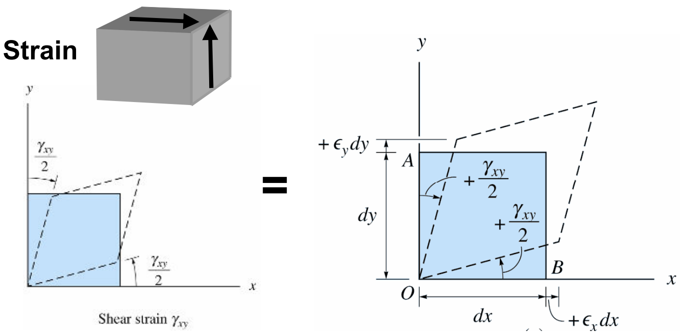
- A structure should be designed so that its material and cross sectional dimensions can resist the maximum normal and shear stresses imposed on it. Equally important also that the structure does not deform much under the load, i.e., the ability to resist strains is crucial to the serviceability of structures.

- Normal Strain (due to axial load + bending moment) and Shear Strain (due to transverse shear + torsion).

Normal Strain



Shear Strain



Strain = ε = Unitless = ΔL / L

Positive Signs
(elongation and angle)

Questions:

Is this the maximum strain? **If not**, then

What is the value of maximum <u>normal</u> strain and the plane in which it exists? and
What is the value of maximum <u>shear</u> strain and the plane in which it exists?

- General equations for strains on a plane at **angle θ** for a member under two dimensional strain. Notice that all equations look the same as those of stress transformation, except that τ_{xy} is resembled by $\frac{\gamma_{xy}}{2}$:

<u>General Equations:</u> Given the three constants ϵ_x, ϵ_y, γ_{xy} then,

Normal strain at any angle θ: $$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

Shear strain at any angle θ: $$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\epsilon_x - \epsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

<u>Principal (Normal) Strain</u>:

Orientation: $$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

Max. Value: $$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Shear strain at this plane: Zero

<u>Maximum Shear Strain</u>:

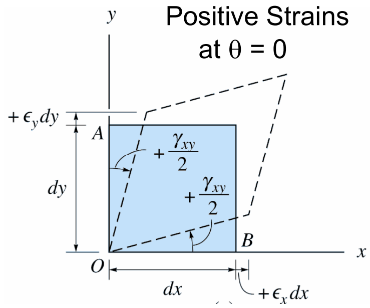
Orientation: $$\tan 2\theta_s = -\left(\frac{\epsilon_x - \epsilon_y}{\gamma_{xy}}\right)$$

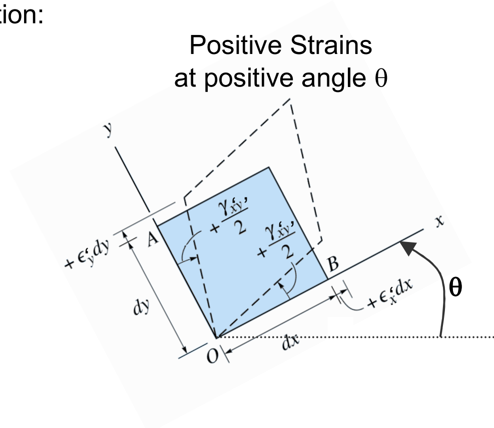
Max. Value: $$\frac{\gamma_{\substack{\max \\ \text{in-plane}}}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

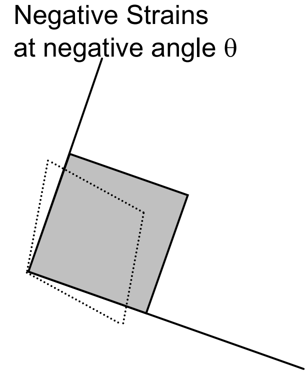
Normal strain at this plane: $$\epsilon_{avg} = \frac{\epsilon_x + \epsilon_y}{2}$$

Solved Problems 10-1 to 10-8

- Strains before and after transformation:







Important Observations:

1. $\varepsilon_x + \varepsilon_y = \varepsilon_{x'} + \varepsilon_{y'}$ = Constant (90 degrees apart) for any orientation.

2. The plane in which shear strain $\gamma_{x'y'} / 2 = 0$ is when:

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\epsilon_x - \epsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta = 0$$

or $\tan 2\theta = \gamma_{xy} / (\varepsilon_x - \varepsilon_y)$ or at θ_1 , θ_2 having 90 degrees apart. These are called **principal planes**.

3. $\varepsilon_{x'}$ becomes maximum when $d\varepsilon_{x'} / d\theta = 0$, or when differentiating the following equation:

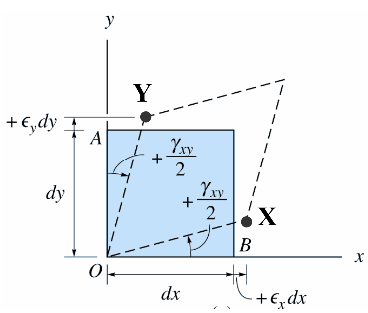
$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

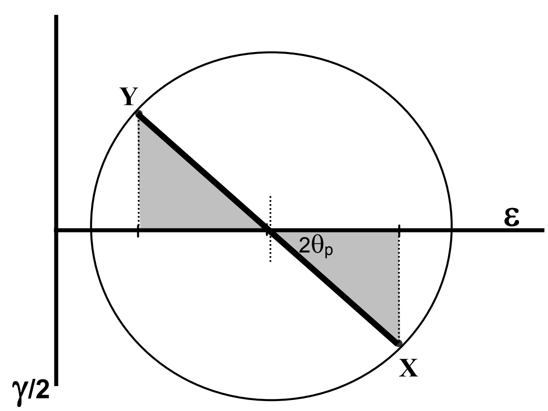
we get, $\tan 2\theta_p = \gamma_{xy} / (\varepsilon_x - \varepsilon_y)$ or, exactly at the principal planes, which has shear strain = 0.

4. $\gamma_{x'y'}$ is maximum when $d\gamma / d\theta = 0$, or when: $\tan 2\theta_s = -(\varepsilon_x - \varepsilon_y) / \gamma_{xy}$

5. Similar to single stress situation $\theta_s = 45^o$ from θ_p.

6. Mohr's circle of strain: **(Shear strain is positive in the bottom half of the circle)**





ε_{min} ? , ε_{max} ? , γ_{max} ? , θ_p ? , θ_s?

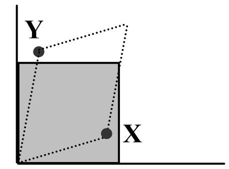
Example:

Given $\varepsilon_{x'} = -200 \times 10^{-6}$, $\varepsilon_{y'} = 1000 \times 10^{-6}$, $\gamma_{xy} = 900 \times 10^{-6}$. Find the strains associated with x'y' axes inclined at 30 degrees clockwise. Find principal strains and the maximum shear strain along with the orientation of elements.

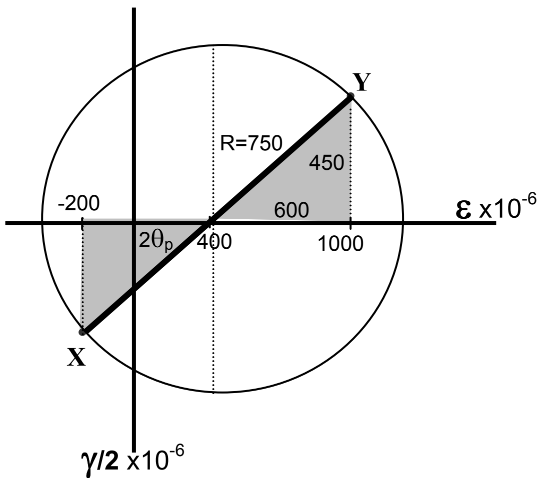
Solution

First, we sketch the element with the given strains, as follows.

Then, we define two points **X** and **Y** to draw Mohr's circle.



Shorter in X
Longer in Y
+ive shear strain.



R = Sqrt (60^2 + 450^2) = 750

Principal Strains:

ε_{max} , ε_{min} = 400 ± 750 = 1150 $\times 10^{-6}$, -350 $\times 10^{-6}$

$\gamma_{x'y'}$ at principal planes = 0

$2\theta_p = \tan^{-1}(450 / 600) = 36.8^o$

Max Shear Strains:

γ_{max} **/ 2** = R = 750 $\times 10^{-6}$

$\varepsilon_{x'} = \varepsilon_{y'}$ at Max shear plane = 400 $\times 10^{-6}$

$2\theta_s = 36.8^o + 90 = 126.8^o$

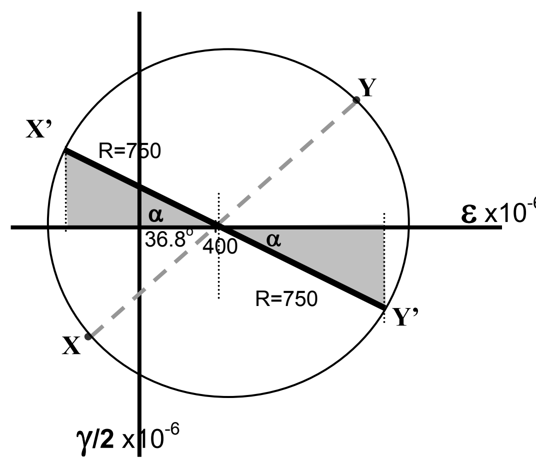
$\alpha = 60 - 36.8 = 23.2$

Then

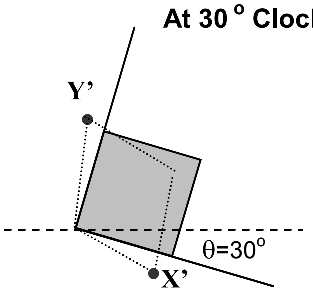
$\varepsilon_{x'} = 400 - R \cos\alpha = 400 - 750 \times \cos 23.2 = -290 \times 10^{-6}$

$\varepsilon_{y'} = 400 + R \cos\alpha = 400 + 750 \times \cos 23.2 = 1090 \times 10^{-6}$

$\gamma_{x'y'} / 2 = R \sin\alpha = -750 \sin 23.2 = -295 \times 10^{-6}$

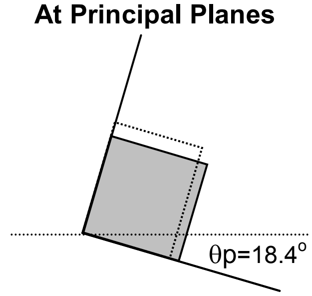


At 30° Clockwise



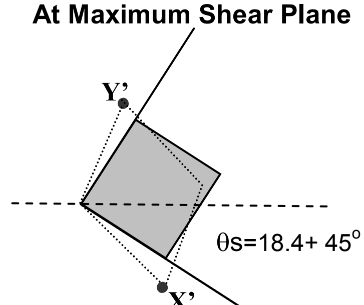
Shorter in X'
Longer in Y'
-ive shear strain (clockwise rotation)

At Principal Planes



Shorter in X'
Longer in Y'
No Shear strain

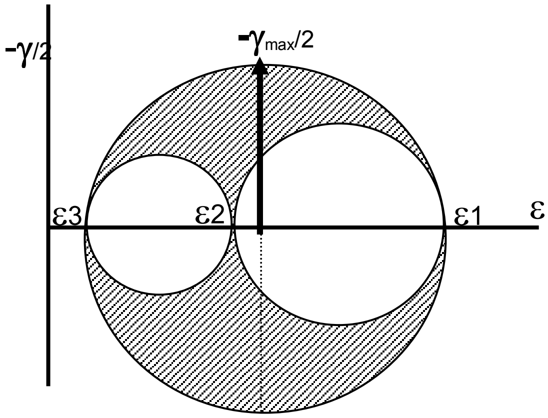
At Maximum Shear Plane



Longer in X'
Longer in Y'
-ive shear strain

Absolute maximum shear strain

Assume $\varepsilon 1 > \varepsilon 2 > \varepsilon 3$ are principal normal strains (no shear), then let's draw Mohr's circle.

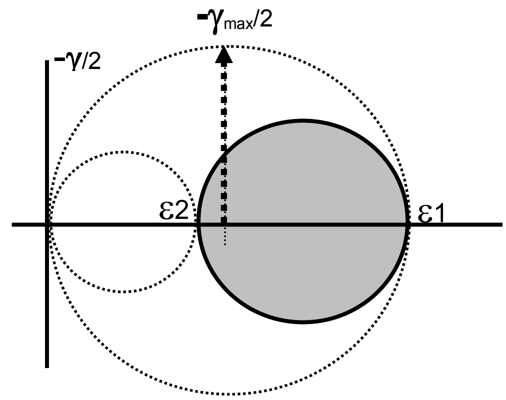


$\gamma_{max}/2 = (\varepsilon_{max} - \varepsilon_{min}) / 2$

Note: Even if $\varepsilon 3 = 0$, 3-D analysis is essential.

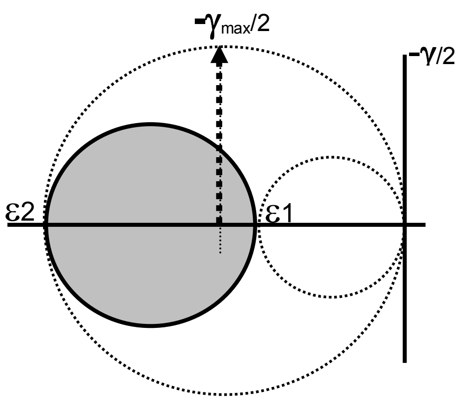
Case 1: both $\varepsilon 1$ and $\varepsilon 2$ are <u>positive</u>

Then, $\gamma_{max}/2 = \varepsilon 1 / 2$



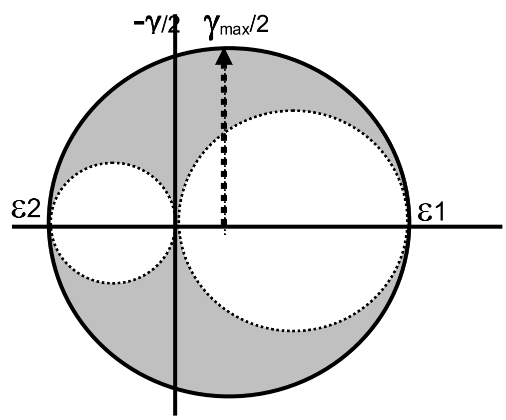
Case 2: both $\varepsilon 1$ and $\varepsilon 2$ are <u>negative</u>

Then, $\gamma_{max}/2 = \varepsilon 2 / 2$



Case 3: $\varepsilon 1$ and $\varepsilon 2$ have opposite signs

Then, $\gamma_{max}/2 = (\varepsilon 1 - \varepsilon 2) / 2$



Solved Examples: 10-1 to 10-7

5. Strain Measurements Using Strain Rosettes

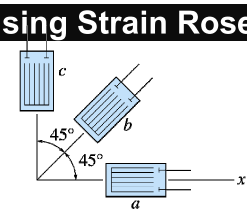
- 45° strain rosette versus 60° strain rosette
- Cemented on surface
- Its electrical resistance changes when wires are stretched or compressed with the material being studied
- Resistance changes are measured and interpreted as changes in deformation
- Three values to get the state of strain at the point

- Automated condition assessment of bridges
- Check the strains on older structures

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

or

$\varepsilon x' = \varepsilon x \cos^2\theta + \varepsilon y \sin^2\theta + \gamma xy \cos\theta \cdot \sin\theta$



Readings: εa, εb, εc

At: θa=0, θb=45, θc=90

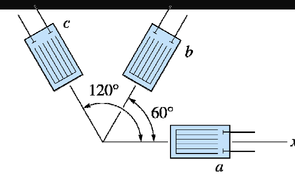
Unknowns: εx, εy, γxy

Applying into the general equation:

$\varepsilon x = \varepsilon a$

$\varepsilon y = \varepsilon c$

$\gamma xy = 2\varepsilon b - (\varepsilon a + \varepsilon c)$



Readings: εa, εb, εc

At: θa=0, θb=60, θc=120

Unknowns: εx, εy, γxy

Applying into the general equation:

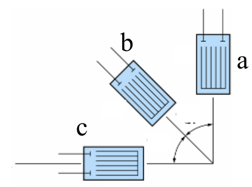
$\varepsilon x = \varepsilon a$

$\varepsilon y = (2\varepsilon b + 2\varepsilon c - \varepsilon a) / 3$

$\gamma xy = 2(\varepsilon b - \varepsilon c) / \text{Sqrt}(3)$

Substitute into either equation 3 times using εa, εb, εc to get the unknowns εx, εy, γxy at the measurement point.

Example:



Using the strain rosette shown, the measured values at each stain gauge is as follows:

$\varepsilon a = 8 \times 10^{-4}$, $\varepsilon b = -6 \times 10^{-4}$, $\varepsilon c = -4 \times 10^{-4}$

Determine the principal strains at the point.

<u>Solution Using Equations</u>:

θa = 90, θb = 135, θc = 180

Applying into the general strain transformation equation:

$\varepsilon a = 8 \times 10^{-4} = \varepsilon x \cos^2 90 + \varepsilon y \sin^2 90 + \gamma xy \cos 90 \cdot \sin 90$

$\varepsilon b = -6 \times 10^{-4} = \varepsilon x \cos^2 135 + \varepsilon y \sin^2 135 + \gamma xy \cos 135 \cdot \sin 135$

$\varepsilon c = -4 \times 10^{-4} = \varepsilon x \cos^2 180 + \varepsilon y \sin^2 180 + \gamma xy \cos 180 \cdot \sin 180$

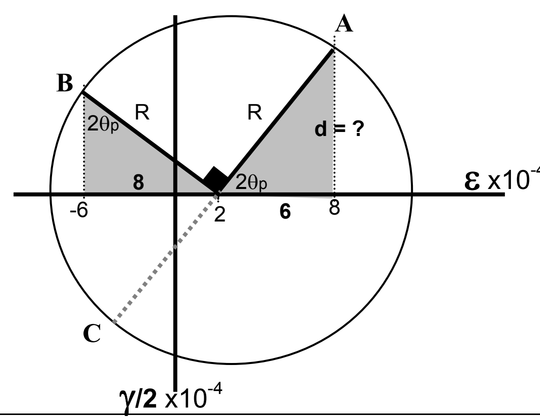
<u>Then:</u>

$\varepsilon y = \varepsilon a = 8 \times 10^{-4}$;

$\varepsilon x = \varepsilon c = -4 \times 10^{-4}$; $\gamma xy = 16 \times 10^{-4}$

Using Mohr's circle, we determine principal strains: <u>$\varepsilon 1 = 12 \times 10^{-4}$; $\varepsilon 2 = -8 \times 10^{-4}$</u>

<u>Solution Using Only Mohr's Circle:</u>



Directions (a) and (c) are 90 degrees apart

This means that the <u>center of the circle</u> is the

Average strain = $(\varepsilon a + \varepsilon c) / 2 = (8 \times 10^{-4} - 4 \times 10^{-4}) / 2 = 2 \times 10^{-4}$

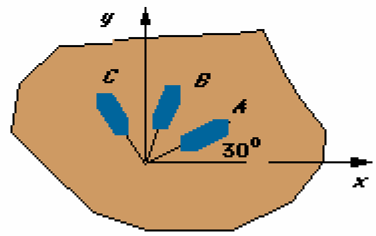
From the two triangles shown, d = ______,

Then, R = Sqrt($d^2 + 6^2$) = __________

As such,

<u>$\varepsilon 1 = 2 + R = 12 \times 10^{-4}$; $\varepsilon 2 = 2 - R = -8 \times 10^{-4}$</u>

Example:



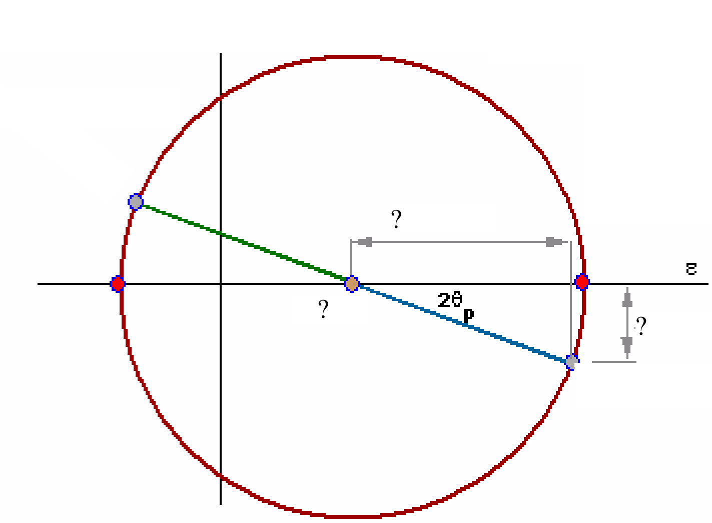
A rectangular strain gage rosette is applied to the surface of a component for which the strains are

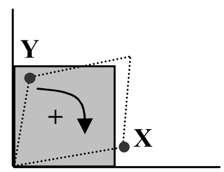
$\varepsilon_x = 6800\ \mu \qquad \varepsilon_y = -1700\ \mu \qquad \gamma_{xy} = 2720\ \mu$

Determine the strain measured by each strain gage.

Solution:

Strategy: We draw a Mohr's circle for strain and on it will find the strains at the orientations of the strain gauges (45° apart).



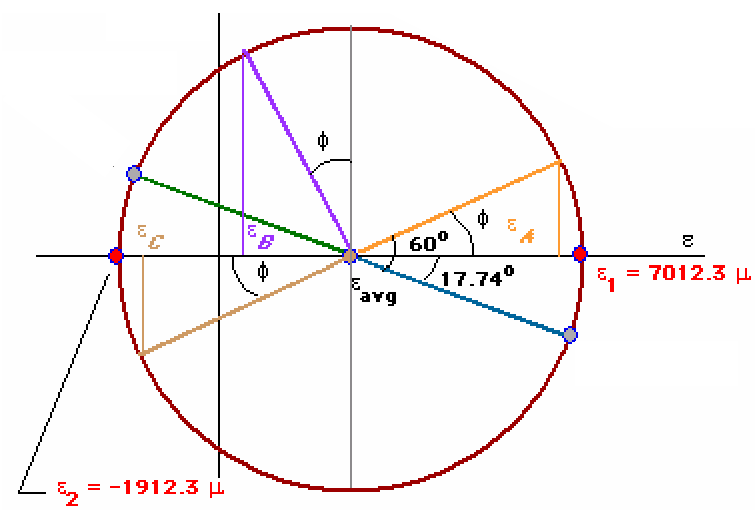


Longer in X
Shorter in Y
+ive shear strain.

$\varepsilon_{avg} = (\varepsilon_x + \varepsilon_y)/2 = (6800 - 1700)/2 = 2550$

$R = [(4250)^2 + (1360)^2]^{1/2} = 4462.3$

$2\theta_p = \tan^{-1}(1360/4250) = 17.74^o$



$\phi = 60^o - 17.74^o = 42.26^o$

$\varepsilon_A = \varepsilon_{avg} + R\cos\phi$

$= 2550 + 4462.3 \cos 42.26^o$

$\varepsilon_A = 5852.8\ \mu$

$\varepsilon_B = \varepsilon_{avg} - R\sin\phi$

$= 2550 - 4462.3 \sin 42.26^o$

$\varepsilon_B = -450.61\ \mu$

$\varepsilon_C = \varepsilon_{avg} - R\cos\phi$

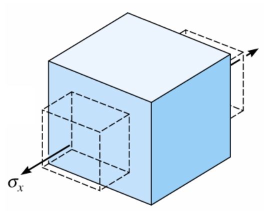
$= 2550 - 4462.3 \cos 42.26^o$

$\varepsilon_C = -752.8\ \mu$

6. Relationship between Stress & Strain (Generalized Hooke's Law)

- A stress in one direction causes elongation in its direction and shortening in the other two depending on the material's Poisson's ratio (ν).

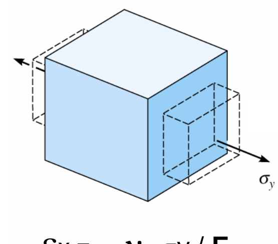
Generalized Hooke's law



$\varepsilon x = \sigma x / \mathbf{E}$

$\varepsilon y = -\nu . \sigma x / \mathbf{E}$

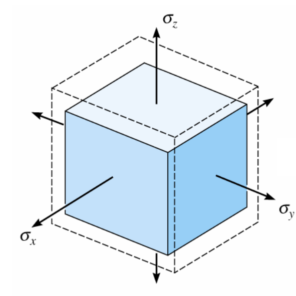
$\varepsilon z = -\nu . \sigma x / \mathbf{E}$



$\varepsilon x = -\nu . \sigma y / \mathbf{E}$

$\varepsilon y = \sigma y / \mathbf{E}$

$\varepsilon z = -\nu . \sigma y / \mathbf{E}$



$\varepsilon x =$

$\varepsilon y =$

$\varepsilon z =$

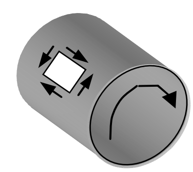
- Assumptions: (1) τ has not correlation with εx and εy; (2) σx and σy have no relation with γxy ; (3) principal strains occur in directions parallel to principal stresses.

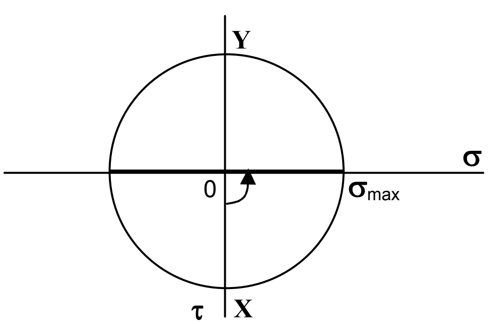
-General Equations:

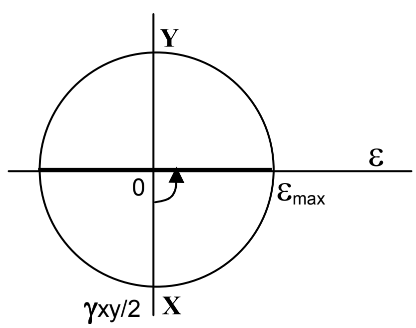
$\mathbf{E}.\varepsilon x = \sigma x - \nu (\sigma y + \sigma z)$;	$\mathbf{G}. \gamma xy = \tau xy$
$\mathbf{E}.\varepsilon y = \sigma y - \nu (\sigma x + \sigma z)$;	$\mathbf{G}. \gamma yz = \tau yz$
$\mathbf{E}.\varepsilon z = \sigma z - \nu (\sigma x + \sigma y)$;	$\mathbf{G}. \gamma zx = \tau zx$

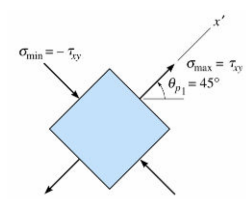
- Relationship between **E**, ν , **G**:

Let's consider the case of pure torsion, i.e., $\sigma_x = 0$ and $\sigma_y = 0$, Let's draw Mohr's circles for both stress and strains.









Principal stresses are: $\sigma_1 = \tau xy$; $\sigma_2 = -\tau xy$

Principal strains are: $\varepsilon_1 = \gamma xy/2$; $\varepsilon_2 = -\gamma xy/2$

Now, let's apply Hook's Equation, as follows:

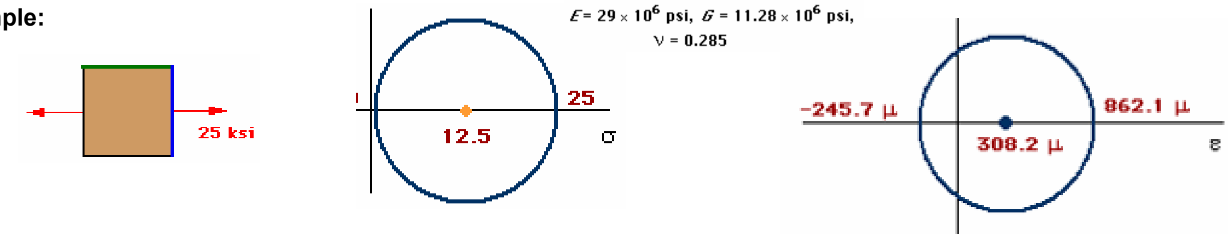
$\mathbf{E}.\varepsilon 1 = \sigma 1 - \nu (\sigma 2)$; then

$\mathbf{E}.\varepsilon 1 = \underline{\mathbf{E}. \gamma xy/2} = \tau xy - \nu (-\tau xy) = \tau xy . (1 + \nu) = \underline{\mathbf{G}. \gamma xy . (1 + \nu)}$

Then $\mathbf{G} = \mathbf{E} / 2 (1 + \nu)$, $\mathbf{K} = \mathbf{E} / 3 (1 - 2 \nu)$ Bulk Modulus

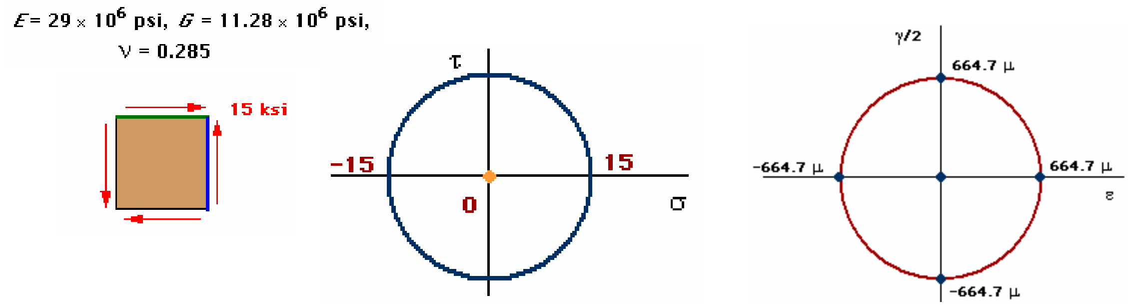
Note: Since most engineering materials has $\nu = 1/3$, then $\mathbf{G = 3/8\ E}$ and $\mathbf{K = E}$

Example:

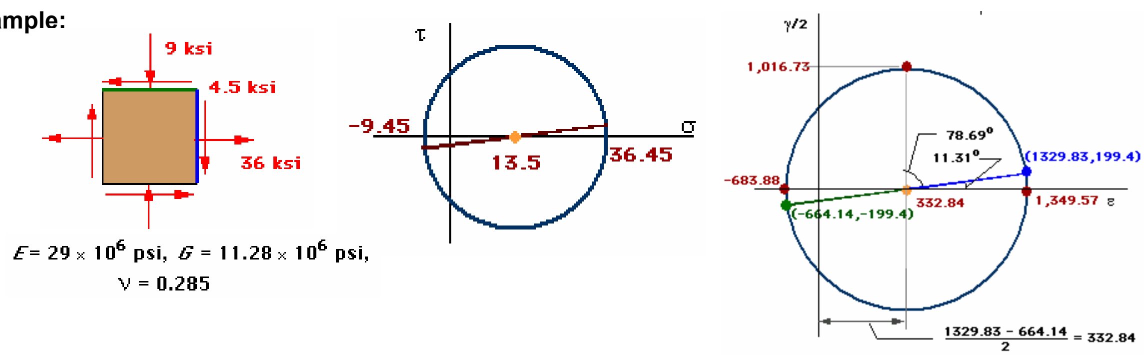


Notice the difference between Mohr's circles for stress & strain

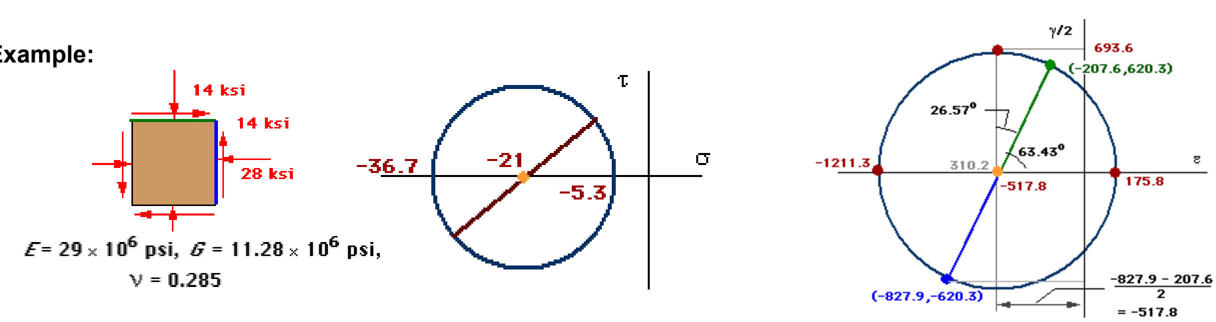
Example:



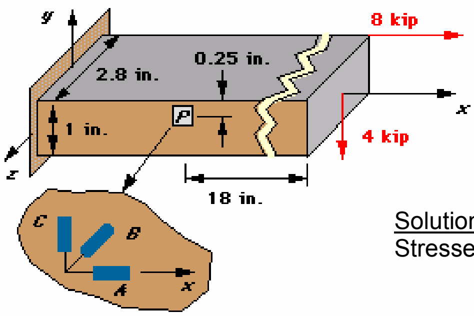
Example:



Example:



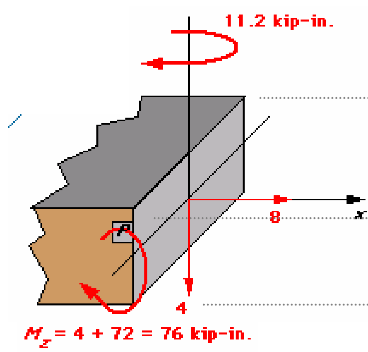
Example



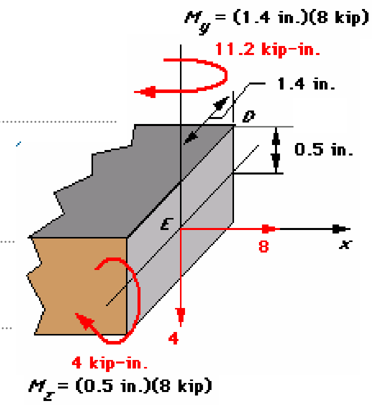
A beam with $E = 10 \times 10^6$ psi and $\nu = 0.33$ is subjected to the loads shown. At point P, a rectangular strain gage rosette is applied. Determine the strain indicated by each of the strain gages.

Solution Approach: Since we are given the forces, let's calculate the Stresses at point P, then, convert these stresses into strains.

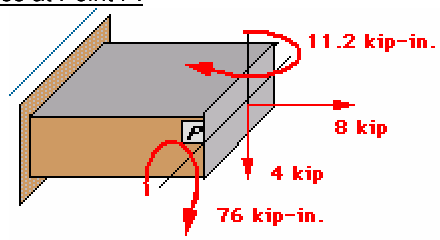
Forces on Section at P.



Forces at end of beam.



Stresses at Point P:



Normal stresses

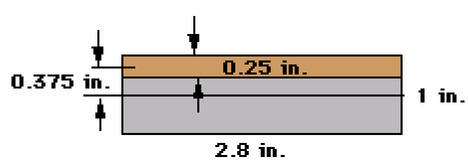
$$\sigma_x = \frac{P}{A} + \frac{M_z y}{I_z} - \frac{M_y z}{I_y}$$

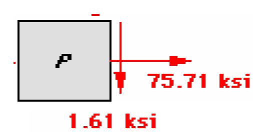
$$= \frac{8}{(1)(2.8)} + \frac{(76)(0.25)}{(2.8)(1^3)/12} - \frac{(11.2)(1.4)}{(1)(2.8^3)/12}$$

$$= 2.86 + 81.43 - 8.57 = 75.71 \text{ ksi}$$

Shear Stresses

The 4-kip force produces a shearing stress at point P.





$$\tau = \frac{VQ}{It_y} = - \frac{(4)[(2.8)(0.25)(0.375)]}{[(2.8)(1^3)/12](2.8)} = 1.61 \text{ ksi}$$

Strains at Point P:

$$\varepsilon_A = \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{10 \times 10^6}(75.71 \times 10^3) = 7{,}571.43\ \mu$$

$$\varepsilon_C = \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{10 \times 10^6}[0 - 0.33(75.71 \times 10^3)] = -2{,}498.57\ \mu$$

$$2\varepsilon_B = \varepsilon_x + \varepsilon_y + \gamma_{xy} \qquad \text{where } \gamma_{xy} = \tau_{xy}/G$$

$$G = \frac{E}{2(1+\nu)} = \frac{10 \times 10^6}{2(1+0.33)} = 3.76 \times 10^6 \text{ psi} \qquad \gamma_{xy} = \frac{-1.61 \times 10^3}{3.76 \times 10^6} = -427.5\ \mu$$

$$\varepsilon_B = \frac{1}{2}(7{,}571.43 - 2{,}498.57 - 427.5) = 2{,}322.68\ \mu$$

Solved Problems 10-9 to 10-11

7. Theories of Failure

Ductile Material (**Yield** Failure)

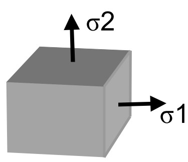
All theories deal with **PRINCIPAL STRESSES**

Brittle Material (**Fracture** Failure)

- Max. normal stress (Rankin's Theory)
- Max. shear stress (Tresca Criterion)
- Max. Energy of Distortion (Von Mises Criterion)
- Other: Max. principal strain (St. Venant)

Max. normal stress (Rankin's Theory)

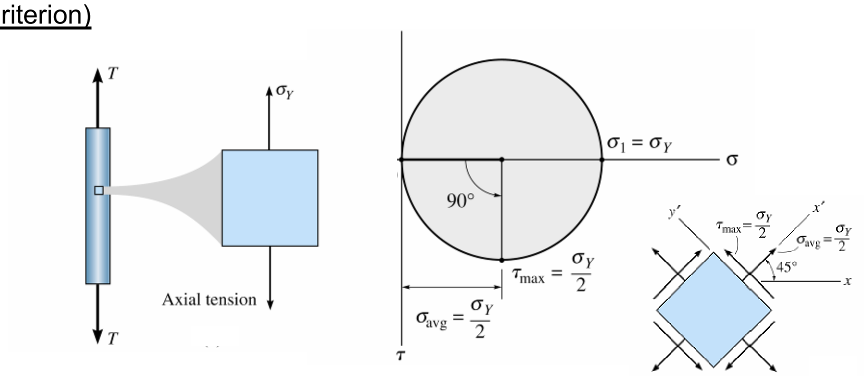
Principal stresses



Failure when: $|\sigma 1| > \sigma y / F.S.$ or
$|\sigma 2| > \sigma y / F.S.$

Max. shear stress (Tresca Criterion)

A specimen under tension reached maximum stress σy, then, the maximum shear that the material can resist is **σy /2** from Mohr's Circle.



Then, failure is when

$$\tau > \sigma y / (2 * F.S.)$$

Absolute max. shear (3-D analysis)

Energy of Distortion (Von Mises Criterion)

To be safe, U_d on element $< U_d$ yield $U = ½\, \sigma.\varepsilon$

For the 3-D stress Case:

$$\frac{1}{12G}\left[(\sigma 1 - \sigma 2)^2 + (\sigma 2 - \sigma 3)^2 + (\sigma 3 - \sigma 1)^2\right] < \frac{2}{12G}\sigma^2_{yield}$$

or Simply, ⇨ $(\sigma 1 - \sigma 2)^2 + (\sigma 2 - \sigma 3)^2 + (\sigma 3 - \sigma 1)^2 < 2\,\sigma^2_{yield}$

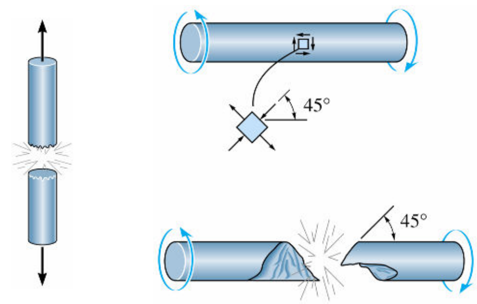
For the 2-D stress Case: ($\sigma 3 = 0$)

⇨ $(\sigma 1^2 - \sigma 1\, \sigma 2 + \sigma 2^2) < \sigma^2_{yield}$

Other: Max. principal strain (St. Venant)   Rarely used

Using Hooke's law $\varepsilon_{max} = [\sigma 1 - \nu (\sigma 2 + \sigma 3)] / E \quad < \quad \sigma_{yield} / E$

Fracture of Brittle Materials



Brittle materials are relatively weak in Tension.

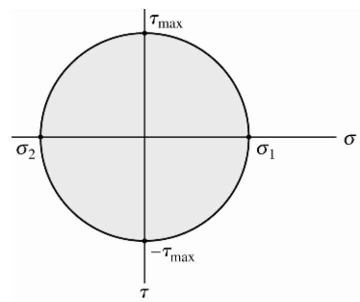
Failure criterion is Maximum Principal Tensile Stress.

Under Tensile force, failure is due to tension.

Under Torsion, failure is still due to tension at an angle.

Element is safe when:

$$|\sigma_1| \leq \sigma_{ult}$$

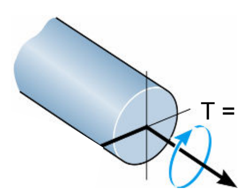


Example: Twist of a piece of chalk.

Solved Examples 10-12 to 10-14

Example: A steel shaft (45 mm in diameter) is exposed to a tensile yield strength = σyield = 250 MPa. Determine P at which yield occurs using Von Mises and Tresca critera.

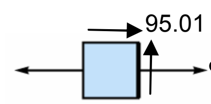
Solution



1) Principal Stresses

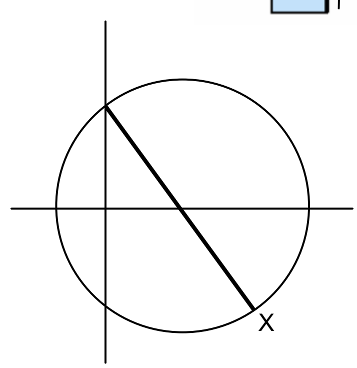
$\sigma x = P / A = P / \pi \, (0.0225)^2$

$\tau xy = T.c / J = 1.7 \times (0.0225) / ½ \, \pi \, (0.0225)^4 = 95.01$



Mohr's circle:

Center = $\sigma x / 2$
R = $[(\sigma x/2)^2 + \tau xy^2]$ ½
$\sigma 1 = \sigma x / 2 + R$; $\sigma 2 = \sigma x / 2 - R$



2) Using Von Mises

$\sigma 1^2 + \sigma 1 \sigma 2 + \sigma 2^2 = \sigma_{yield}{}^2$

$(\sigma x / 2 + R)^2 + (\sigma x / 2 + R)(\sigma x / 2 - R) + (\sigma x / 2 - R)^2 = \sigma_{yield}{}^2$

$(\sigma x/2)^2 + 3 R^2 = \sigma_{yield}{}^2$, substituting with R,

$\sigma x^2 + 3 \tau xy^2 = \sigma_{yield}{}^2$, substituting with σx & τxy,

$[P / \pi (0.0225)^2]^2 + 3 \times (95.01)^2 = \sigma_{yield}{}^2 = 250^2$

then, P = 299.3 KN

3) Using TRESCA

$\sigma 1$ and $\sigma 2$ have opposite signs, then

$\tau_{max\ (3\text{-}D)} = |\sigma 1 - \sigma 2| / 2$, which reaches failure of $\tau_{yield} = \sigma_{yield} / 2$

$|(\sigma x/2 + R) - (\sigma x/2 - R)| / 2 = \sigma_{yield} / 2$

then, $R = \sigma_{yield} / 2$, substituting with R and squaring both sides,

$(\sigma x/2)^2 + \tau xy^2 = (\sigma_{yield} / 2)^2$, substituting with σx & τxy,

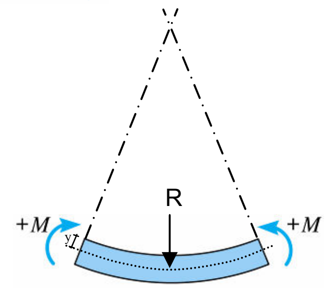
$[P / 2 \pi (0.0225)^2]^2 + (95.01)^2 = 125^2$

then, P = 258.4 KN

Notice the force P under TRESCA (focuses on Shear) is smaller than Von Mises

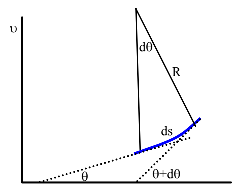
8. Deflection Using the Integration Method

Beams and shafts deflect under load. For serviceability, we need to make sure deflection is within allowable values. Also, the shape of the beam under the load (elastic curve) needs to be studied.



Terminology:

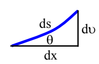
- **EI** = Flexture rigidity or Bending Stiffness
- **R** = Radius of Curvature
- **1/R** = Curvature
- Hookes Law: **<u>1/R = M / EI</u>**
- The elastic curve:



$R d\theta = ds \cong dx$

or

$1/R = d\theta/dx$



Also, $d\upsilon/dx = \tan\theta \cong \theta$

Differentiating both sides, then $d^2\upsilon/dx^2 = d\theta/dx$

Accordingly,

$$\frac{1}{R} = \frac{M}{EI} = \frac{d\theta}{dx} = \frac{d^2\upsilon}{dx^2}$$

Notes:

- Integration of (M/EI) determines the slope of the elastic curve:

$$EI\frac{dy}{dx} = EI\theta = \int_0^x M(x)\,dx + C_1$$

- Double integration of M/EI determines the deflection:

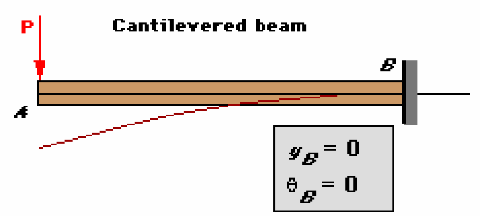
$$EIy = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

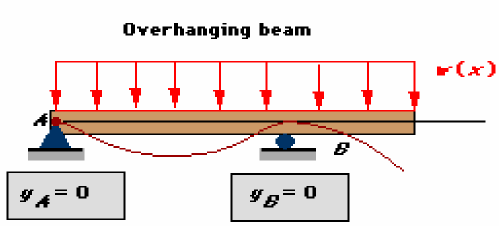
- Recall relationships between load, shear, and bending moment. Now, we can expand it to:

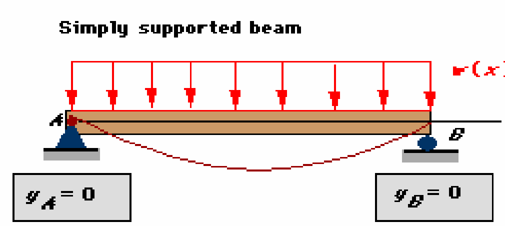
$EI\, d^2\upsilon/dx^2 = M(x)$; $EI\, d^3\upsilon/dx^3 = V(x)$; $EI\, d^4\upsilon/dx^4 = -W(x)$

Determining the integration constants C1 and C2:

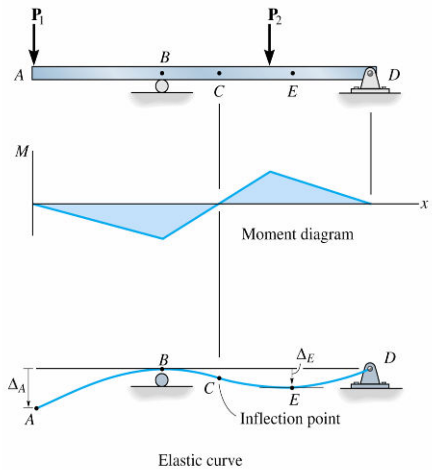
Substituting at points of known deflection and/or slope, we can determine the constants of integration.

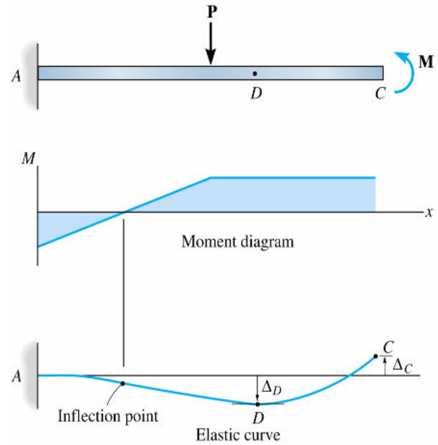


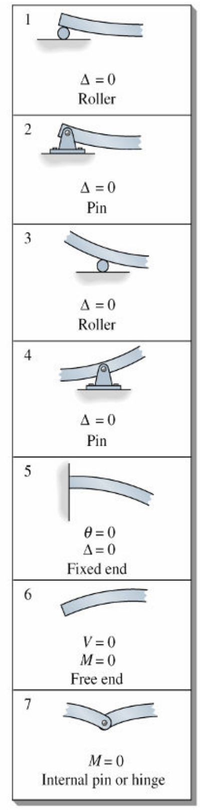




Shape of Elastic Curve: inflection point at location where moment=0







Calculating Slope & Displacement by Integration:

Step-by-Step

1. Get beam reactions: $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma M = 0$
2. Get equation of B.M. at each beam segment with change in load or shape
3. Integrate the moment once to get the slope

$$EI\,\theta(x) = \int_0^x M(x)\,dx + C_1$$

4. Integrate the moment a second time to get the deflection (elastic curve)

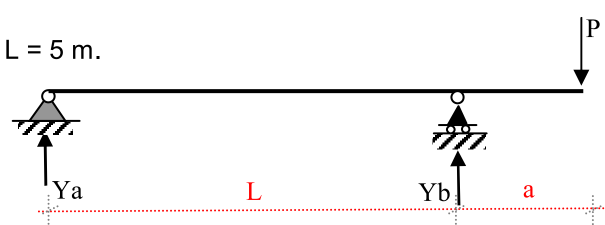
$$EI\,y(x) = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

5. Substitute at points of special conditions (boundary conditions) to get the constants **C1** & **C2**
6. Rewrite the slope and deflection equations using the constants
7. Put slope = 0 to determine the location (x) that has maximum deflection

Example:

For the part AB, determine the equation of the elastic curve and maximum deflection if:

I = 301×10^6 mm^4, E=200 GPa, P=250 KN, a = 1.2 m, L = 5 m.



Solution

1. Reactions:

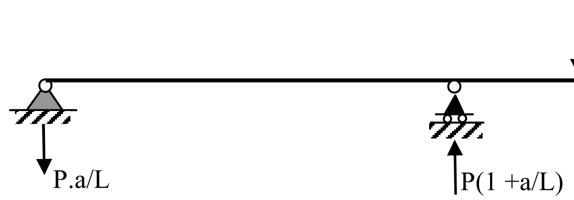
$\circlearrowright^{+} \sum M_a = 0$

$Yb \cdot L - P \cdot (L + a) = 0$

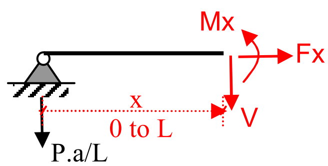
or $\underline{Yb = P\,(1 + a/L)}$

$\uparrow^{+} \sum Y = 0$, then $Ya + Yb - P = 0$ or

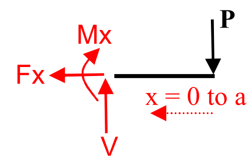
$\underline{Ya = -P \cdot a/L}$



2. Bending moment equations:



$Mx = -P.a.x / L$



$Mx = -P.x$

3. Integrate the moment to get the slope:

$EI\theta(x) = -P.a.x^2 / 2L + C1$

4. Integrate a second time to get the (elastic curve)

$EI\,y(x) = -P.a.x^3 / 6L + C1.x + C2$

5. Substitute at points of known conditions

at support A: [x = 0 . y = 0]
then, $\underline{C2 = 0}$

also, at support B: [x = L , y = 0]
then, $0 = -P\,a\,L^3 / 6L + C1.L$
or $C1 = P.a.L/6$

6. Final equations:

$EI\theta(x) = -P.a.x^2 / 2L + P.a.L/6$ ……(1)

$EI\,y(x) = -P.a.x^3 / 6L + P.a.L.x/6$ ……(2)

7. Put slope = 0 at maximum deflection

$0 = -P.a.x^2 / 2L + P.a.L/6$ get $\underline{x = 0.577L}$

Using this value in equation (2), we get

Max deflection = <u>8 mm Up.</u>

8. Applying same steps at the free end:

$EI\theta(x) = -P.x^2/2 + C3$ …(3)

$EI\,y(x) = -P.x^3 / 6 + C3.x + C4$ ….(4)

Slope at B right = Slope at B Left

Slope left = using equation <u>(1)</u>, x=L

$= -P.a.L/2 + P.a.L/6$

Slope right = using eq. (3), x=a

$= -P.a^2/2 + C3$

we get <u>C3</u>

Also, at B: [x = a , y = 0]

Using Equ. (4), we get <u>C4</u>

Solved problems 12-1 to 12-4

9. Calculating Slope & Displacement by Moment Area Method:

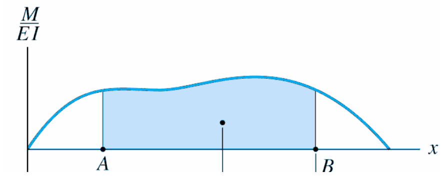
1st Moment Area Theorem:

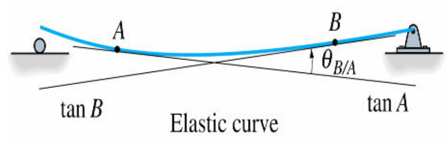
Recall $\frac{M}{EI} = \frac{d\theta}{dx}$

Then,

$$\theta_{B/A} = \int_{X_A}^{X_B} \frac{M}{EI}\, dx$$

change in slope — area under M/EI diagram



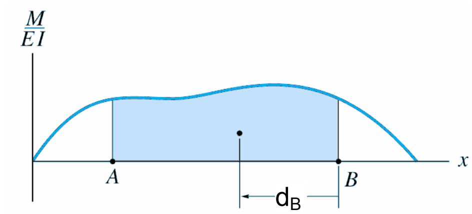


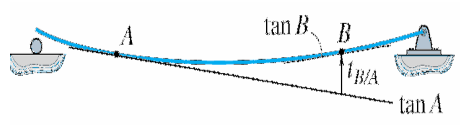
2nd Moment Area Theorem:

t_{BA} = (**vertical** distance from tangent at A to point B on elastic curve)

= Moment of the area under M/EI around point B.

$$= d_B \cdot \int_{X_A}^{X_B} \frac{M}{EI}\, dx$$





Note: $t_{AB} \neq t_{BA}$

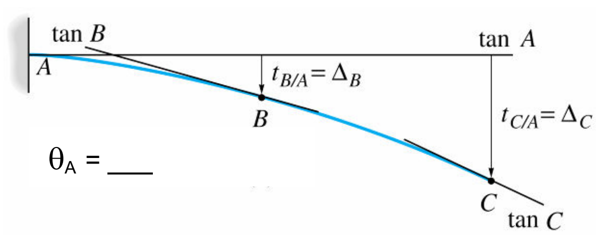
Case 1: **Cantilever**

Notice that tangent at point A is horizontal.

-Deflection at any point: ______________

-Slope at any point:

$$\theta_{B/A} = \int_{X_A}^{X_B} \frac{M}{EI}\, dx = \theta_B - \theta_A = \theta_B$$



θ_A = ___

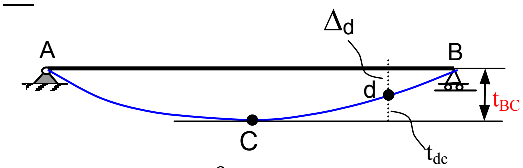
Case 2: **Symmetric Loading – Option 1**

Deflection is max at mid beam (C). At this point θ_C = __

-Deflection at any point: ______________

-Slope at any point:

$$\theta_{B/A} = \int_{X_C}^{X_d} \frac{M}{EI}\, dx = \theta_D - \theta_C = \theta_D$$

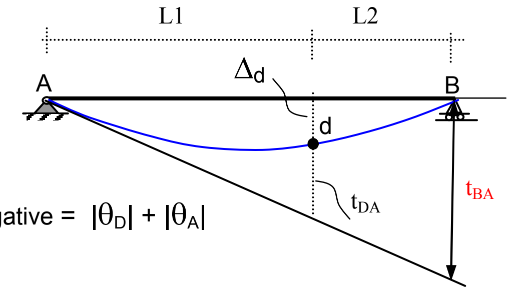


θ_C = __

Δ_C = __

Case 3: **Unsymmetrical Loading – Option 2**

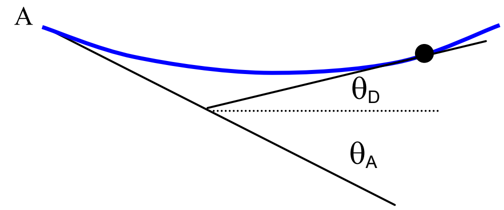
-Deflection at any point: $\Delta_d + t_{DA} = t_{BA} \cdot L1/(L1+L2)$

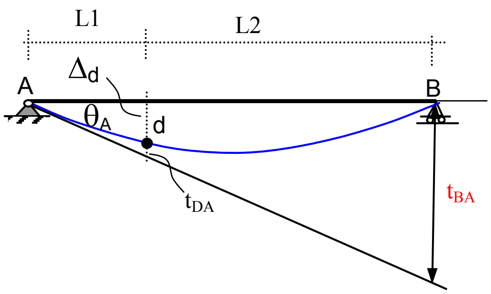


-Slope at any point to the right:

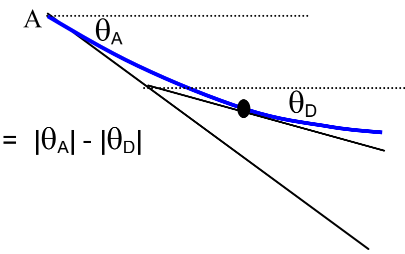
$$\theta_{D/A} = \int_{X_a}^{X_d} \frac{M}{EI}\,dx = \theta_D - \theta_A \text{ with } \theta_A \text{ being negative} = |\theta_D| + |\theta_A|$$

($\theta_A = t_{BA} / (L1+L2)$)





-Slope at any point to the left:



$$\theta_{D/A} = \int_{X_a}^{X_d} \frac{M}{EI}\,dx = \theta_D - \theta_A \text{ , both negative } = |\theta_A| - |\theta_D|$$

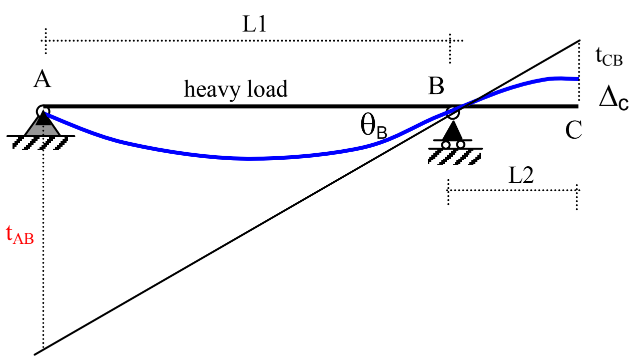
($\theta_A = t_{BA} / (L1+L2)$)

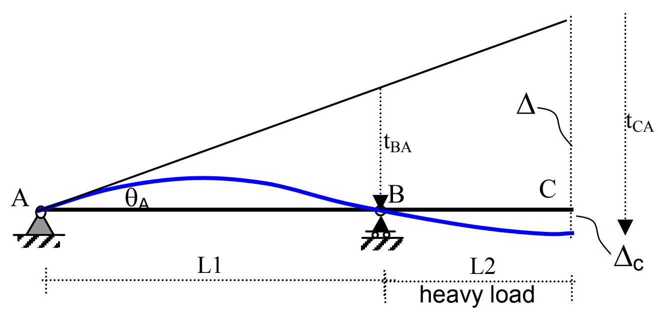
Case 4: **Over-Hanging Beam**

$\theta_B = t_{AB} / L1$

$= (\Delta_C + |t_{CB}|) / L2$

Then, $\Delta_C = |\theta_B \cdot L2| - |t_{CB}|$





$\Delta = t_{BA} \cdot (L1+L2)/L1$

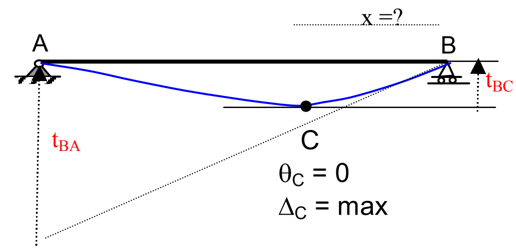
Then, $\Delta_C = |t_{CA}| - |\Delta|$

Case 5: **Unsymmetrical Loading – Point of Max. Deflection**

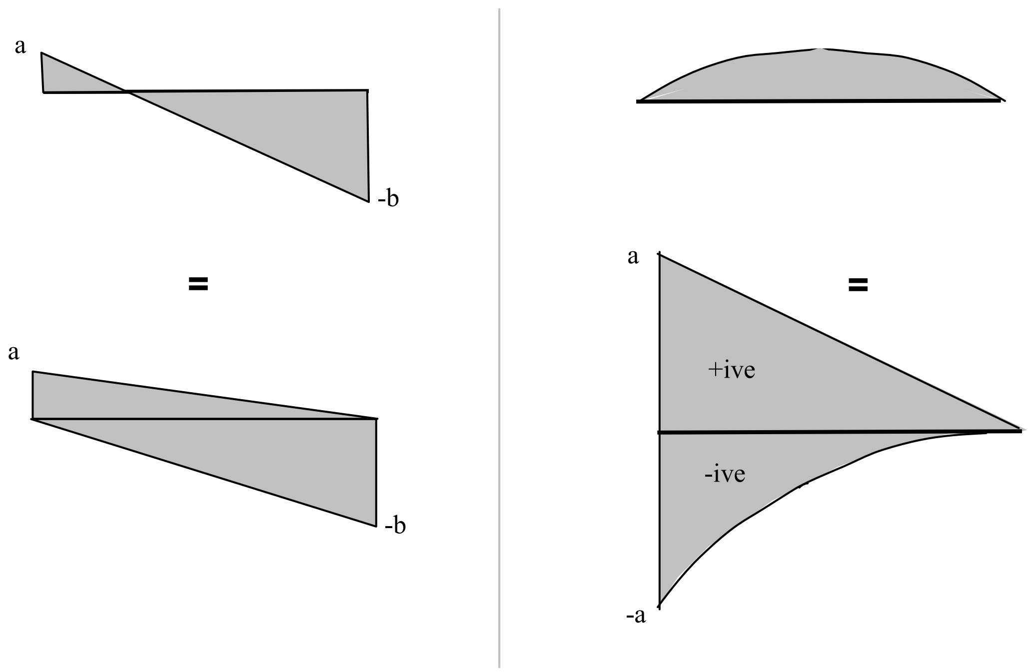
$\theta_B = t_{AB} / L$

$$\theta_{B/C} = \int_{X_C}^{X_B} \frac{M}{EI}\,dx = \theta_B - \theta_C = \theta_B = t_{AB} / L$$

We get x, then $\Delta_C = \max = t_{BC}$



Note: Equivalence in Bending Moment Diagrams

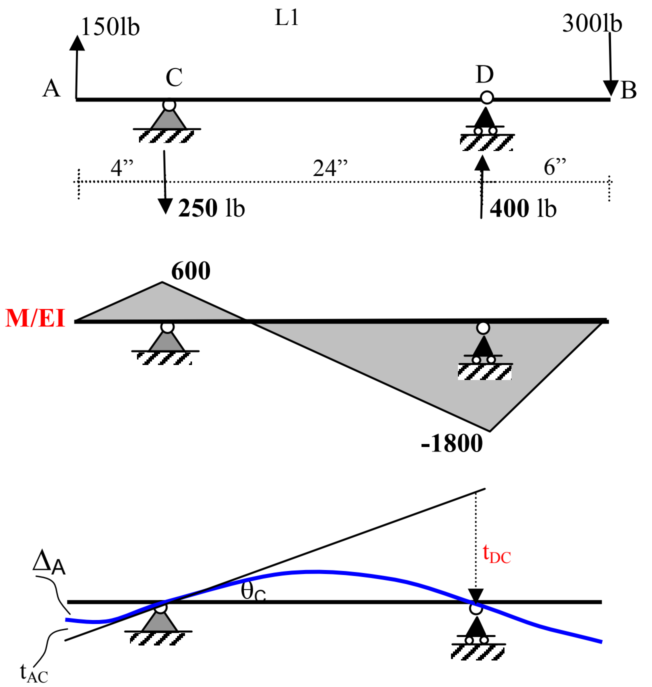


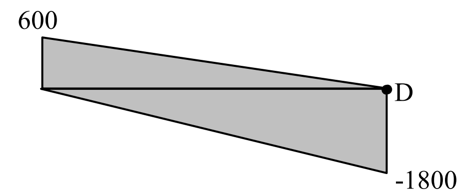
Method of Superposition:

- Using Standard tables for various beam conditions and types of loads (**Appendix C**)
- Adding up deflections caused by individual loads

Solved Problems: 12-7 to 12-15

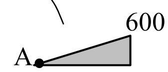
Example: Determine θ_C and Δ_A





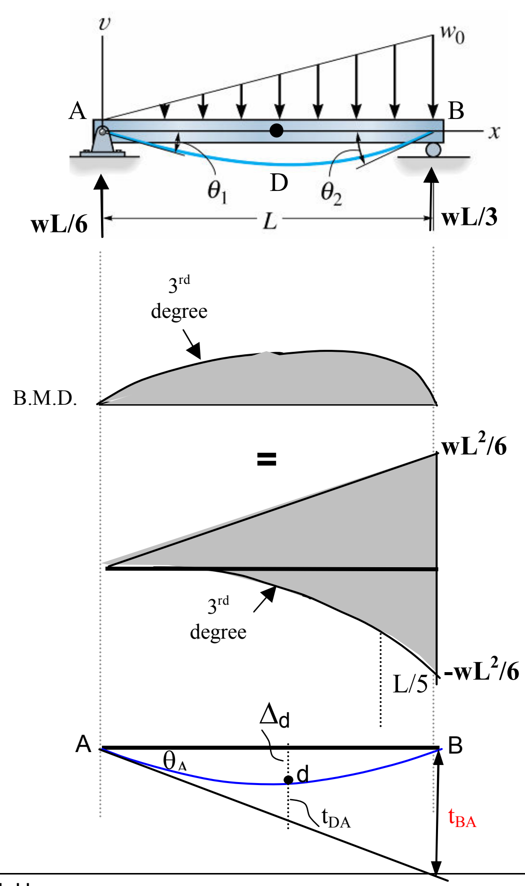
t_{DC} =1/EI [+ (600 x 24 /2) . 2/3 . 24
- (1800 x 24 /2) . 1/3 . 24]

$\theta_C = t_{DC} / 24$
$= (|\Delta_A| + | t_{AC} |) / 4$



t_{AC} = 1/EI [600. 4/2 . 2/3 . 4]

Example: Determine θ_A and Δ_D



$\theta_A = t_{BA} / L$

= Moment of M/EI @ B / L

$= [w.L^2 /6EI . L/2 . L/3 - w. L^2/6EI . L/4 . L/5] / L$

$= 7 w.L^3 / 360EI$

Also, $\Delta_d + t_{DA} = t_{BA} / 2$

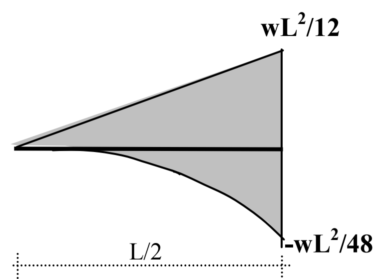
Then

$\Delta_d = | t_{BA} / 2| - | t_{DA} |$

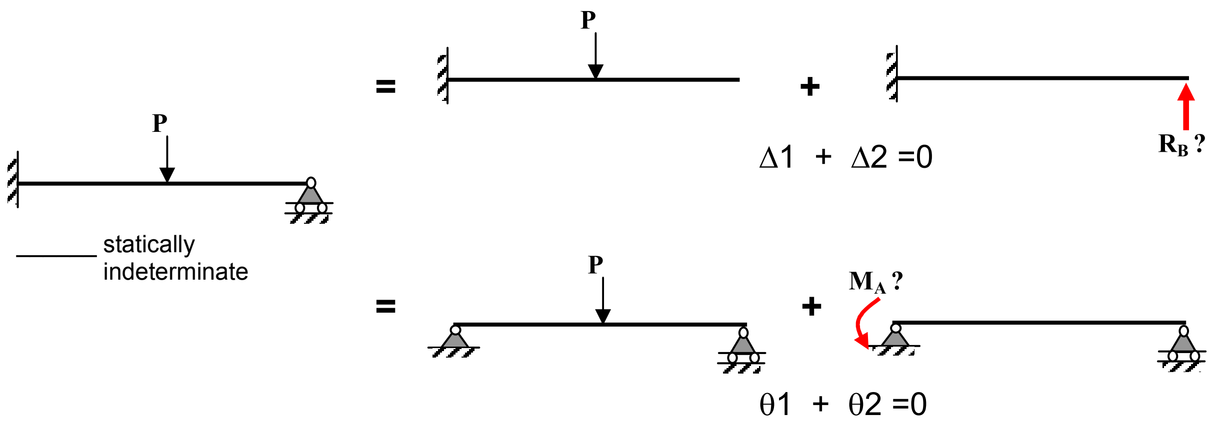
$= 7 w.L^3/720EI$

$- [1/2 .w.L^2 /12EI .L/2 .L/6 - ¼ . wL^2/48EI. L/10]$

$= 5 w.L^4 / 768EI$

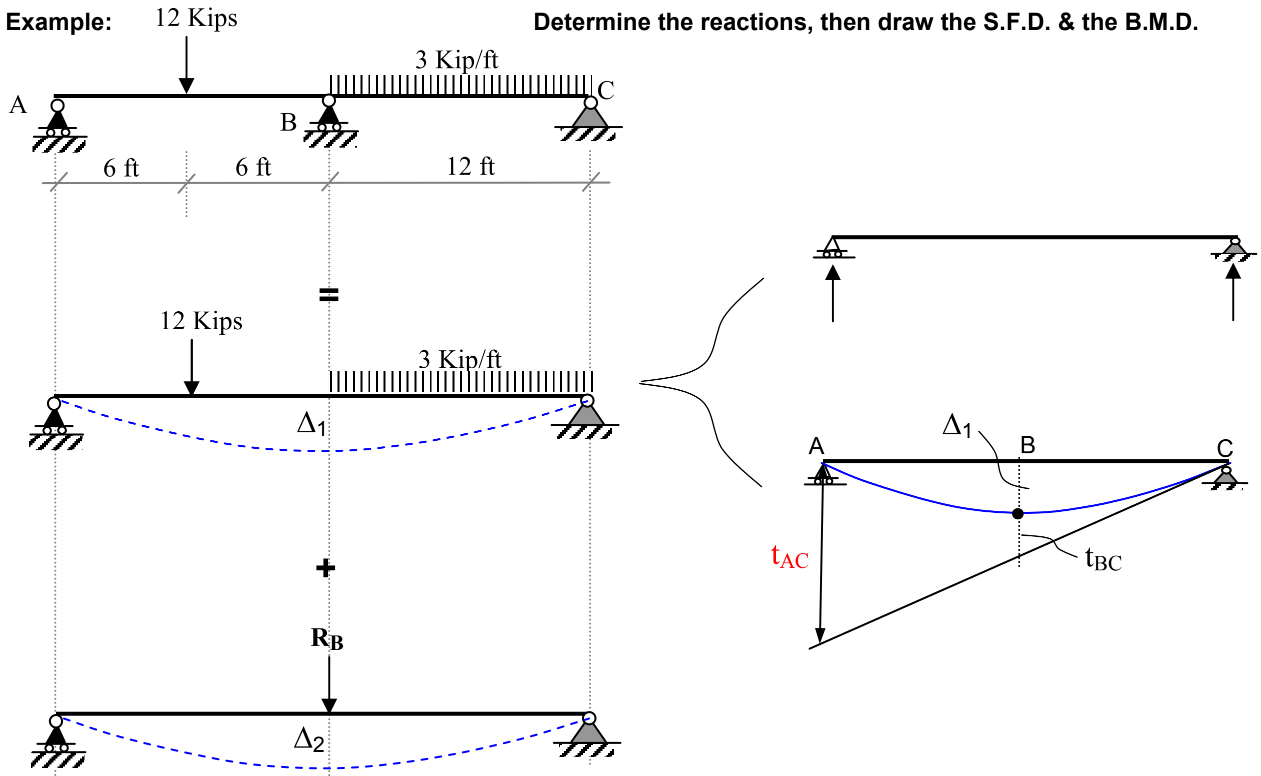


Using Deflection Calculations to Solve Statically Indeterminate Beams



First, we reduce the beam to a statically determinate, then
We compensate for the change in the deflection behavior.

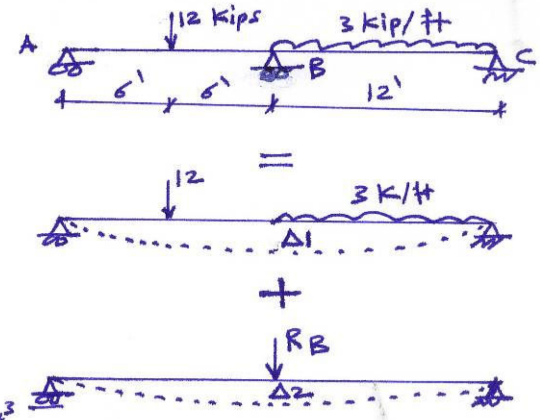
Example: **Determine the reactions, then draw the S.F.D. & the B.M.D.**



$\Delta 2 = -R_B L^3 / 48 EI$

$\Delta 1 + \Delta 2 = 0$

Example: Determine the reactions, then draw the S.F.D. and B.M.D.

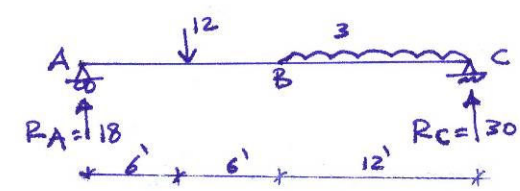


- As shown to the right, the structure is statically indeterminate the middle roller is then removed and a reaction R_B is applied so that $\Delta_1 + \Delta_2 = 0$

Detailed calculations below

from standard tables,

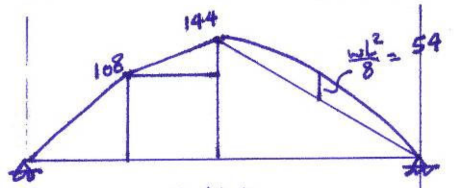
$$\Delta_2 = \frac{PL^3}{48EI} = \frac{R_B \cdot 24^3}{48EI}$$



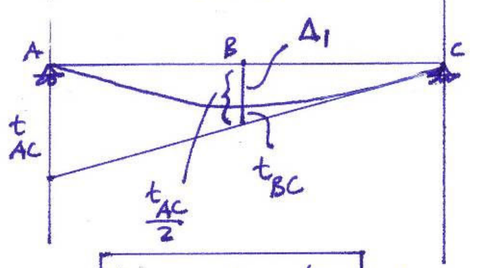
$\circlearrowleft \sum M_A = 0, \quad -12 \times 6 - 36 \times 18 + R_C \cdot 24 = 0$

$R_C = 30$

$+\uparrow \sum F_y = 0 \quad R_A = 18$



B.M.D.



Moment Area Calculations

$$\boxed{\frac{t_{AC}}{2} = \Delta_1 + t_{BC}}, \text{ where}$$

$$t_{AC} = \left[\frac{108 \times 6}{2} \times \frac{2}{3} \times 6 + 108 \times 6 \times 9 + 36 \times \frac{6}{2} \times 10 + \frac{144 \times 12}{2} \times 16 + \frac{2}{3} \times 54 \times 12 \times 18\right] / EI$$

$$= 29808 / EI$$

$$t_{BC} = \left[\frac{144 \times 12}{2} \times 4 + \frac{2}{3} \times 54 \times 12 \times 6\right] / EI$$

$$= 6048 / EI$$

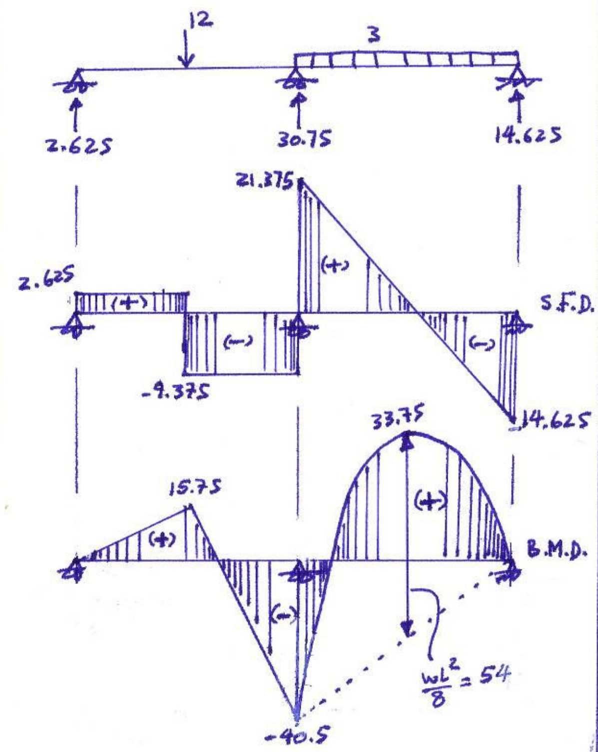
Then, $\Delta_1 = \frac{29808}{2EI} - \frac{6048}{EI} = 8856/EI$

Finally, $\Delta_1 + \Delta_2 = 0 = \frac{8856}{EI} + \frac{R_B \cdot 24^3}{48EI} = 0$

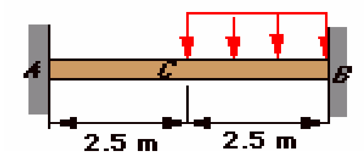
Then $R_B = 30.75$ Kips. ↑

Once $R_B = 30.75$ Kips ↑ was calculated, we calculate remaining reactions

$R_A = 2.625$ Kips ↑ & $R_C = 14.625$ Kips

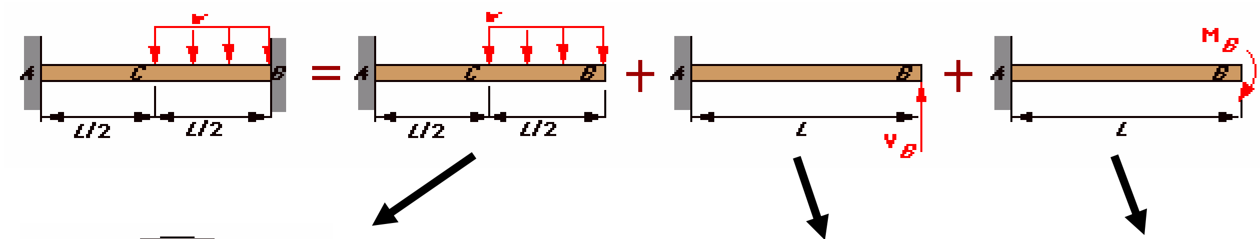


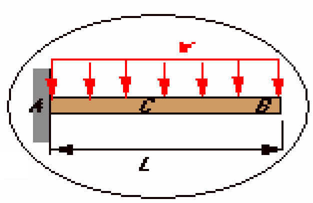
Example:



In this example, we want to determine the reactions at end B of the beam shown knowing that E = 210 GPa and I = 40.8 × 10^6 mm^4.

Solution:

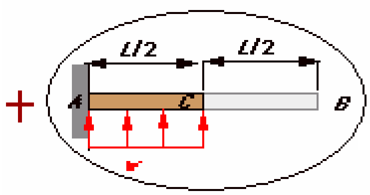




$$\theta_B = -\frac{wL^3}{6EI} \qquad y_B = -\frac{wL^4}{8EI}$$

$$\theta_B = \frac{V_B L^2}{2EI} \qquad y_B = \frac{V_B L^3}{3EI}$$

$$\theta_B = \frac{M_B L}{EI} \qquad y_B = -\frac{M_B L^2}{2EI}$$



$$\theta_B = \theta_C = -\frac{-w(L/2)^3}{6EI} = \frac{wL^3}{48EI}$$

$$y_B = -\frac{-w(L/2)^4}{8EI} + (L/2)\theta_B = \frac{7wL^4}{384EI}$$

$$\theta_B = -\frac{7wL^3}{48EI}$$

$$y_B = -\frac{41wL^4}{384EI}$$

Next we add the slopes and deflections from each model and satisfy the boundary conditions at end B.

$$\theta_B = 0 = -\frac{7wL^3}{48EI} + \frac{V_B L^2}{2EI} - \frac{M_B L}{EI} \qquad M_B = \frac{V_B L}{2} - \frac{7}{48}wL^2$$

$$y_B = 0 = -\frac{41wL^4}{384EI} + \frac{V_B L^3}{3EI} - \frac{M_B L^2}{2EI} \qquad M_B = \frac{2V_B L}{3} - \frac{42}{384}wL^2$$

Equating these two expressions for M_B we can solve for V_B, then M_B.

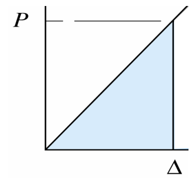
$$\frac{V_B L}{2} - \frac{7}{48}wL^2 = \frac{2V_B L}{3} - \frac{42}{384}wL^2 \qquad V_B = \frac{13}{32}wL \qquad M_B = \frac{11}{192}wL^2$$

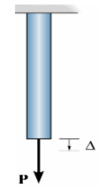
V_B = 36.56 kN $\qquad$ M_B = 25.78 kN-m

10. Strain Energy Method

- For a structural element under load and deformation, External Work $\mathbf{U_e}$ = Internal Strain Energy $\mathbf{U_i}$.

- External work $\mathbf{U_e}$ is a function of the load **P** and deflection Δ. (deflection is at same point and direction of load)

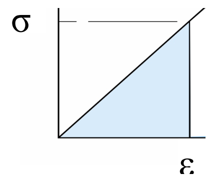




$$U_e = ½\, P.\, \Delta$$

- Also, the Internal strain energy in the structure $\mathbf{U_i}$ is a function of the stress σ and strain ε in the element, summed over the volume of the structure.

$$U_i = \underbrace{½\ \sigma\ .\ \varepsilon}_{\text{Strain Energy per unit volume}}\ .\ V = \frac{\sigma^2}{2E}\ .\ V$$



Normal

$$U_i = \int_V \frac{\sigma^2}{2\,E}\,.\,dV$$

and

Shear

$$U_i = \int_V \frac{\tau^2}{2\,G}\,.\,dV$$

$$V = \int_V dV = \int_A dA \int_0^L dx$$

when A is constant, $V = A \int_0^L dx$

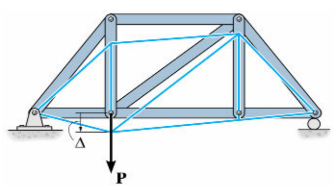
Observe the units.

Strain Energy calculations for different loading conditions are shown in next page.

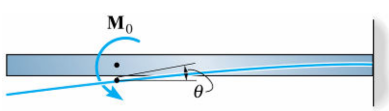
Determining Deflections Using Conservation of Energy

Single External load
Deflection in the direction of load:

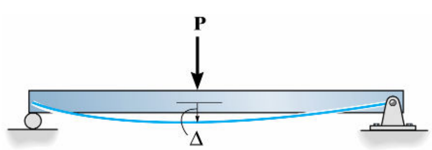
$$U_e = U_i$$



$U_e = ½\, P.\, \Delta$ & $U_i = \sum \frac{N^2 L}{2AE}$



$U_e = ½\, M_o\, .\, \theta$ & $U_i = \int_0^L \frac{f_s V^2\, dx}{2GA} + \int_0^L \frac{M^2\, dx}{2EI}$



$U_e = ½\, P.\, \Delta$ & $U_i = \int_0^L \frac{f_s V^2\, dx}{2GA} + \int_0^L \frac{M^2\, dx}{2EI}$

Limitations: Applies to single load only. Also, in case 2, only solpe is calculated not deflection. Also, how to get deflection at a point at which no direct load is applied.

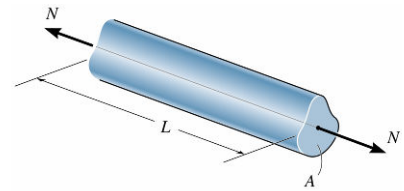
Solved Examples 14-1 to 14-7

Strain Energy Calculations

Normal Stress

Axial Load

$$U_i = \frac{\sigma^2}{2E} \cdot \int_V dV = \frac{N^2}{2EA^2} \int_A dA \int_0^L dx$$



Example: Truss with varying axial loads on individual members.
(Cross section area A is constant, then **V = A . L**)

$$U_i = \frac{N^2 L}{2AE}$$

Bending Moment

$$U_i = \sigma^2/2E \int_V dV \quad \text{or} \quad U_i = \frac{M^2}{2 E I^2} \cdot \underbrace{y^2 \int_A dA}_{= I} \int_0^L dx$$

$$\sigma = \frac{M.y}{I}$$

$$U_i = \int_0^L \frac{M^2}{2 E I} dx$$

Shear Stress

Pure Shear

$$U_i = ½ \, \tau \cdot \gamma \cdot \int_V dV = \tau^2/2G \int_0^L A. dx$$

$$\tau = \frac{V.Q}{I.t}$$

$$U_i = \int_0^L \frac{f_s V^2}{2 G A} dx$$

where, f_s = **6/5** - rectangular section

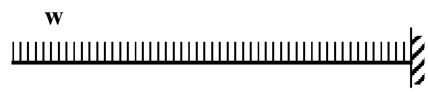
Torsion

$$U_i = ½ \, \tau \cdot \gamma \int_V dV = \tau^2/2G \cdot \int_A dA \cdot \int_0^L dx$$

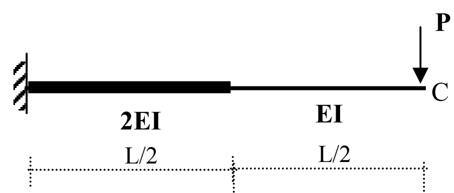
$$\tau = \frac{T.c}{J}$$

$$U_i = \int_0^L \frac{T^2}{2 G J} dx$$

Example: Determine the strain energy due to both shear and bending moment in the following cantilever. The cross section is a square of length a, with EI being constant.

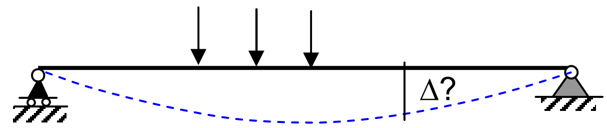


Example: Determine deflection at C, neglect shear strain energy.
.



11. Principle of Virtual Work

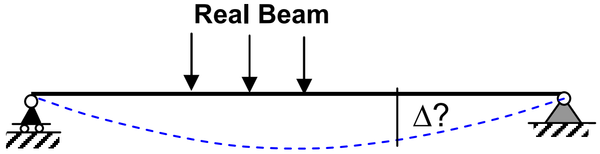
Conservation of Virtual Work



Work-Energy method is not able to determine the deflection at a point at which no direct load exists.

Solution: Put a **virtual load of 1.0** at the desired point of a virtual system. Then apply the principal of conservation of virtual work, as follows:

Real Beam



Virtual Beam

External **Virual** WorK = Internal **Virtual** Energy

½ Virtual load x Real displacement = ½ Virtual Stress x Real Strain x Volume

$$1.0 \times \Delta = \sigma_V \, . \, \varepsilon_R \, . \, V = \sigma_V \, . \, \frac{\sigma_R}{E} \, . \int_A dA \int_0^L dx$$

$$= \frac{n}{A} \, \frac{N}{AE} \, A \int_0^L dx = \int_0^L \frac{n\,N}{A\,E} \, dx \qquad \text{Axial Load}$$

$$+ \int_0^L \frac{m\,M}{E\,I} \, dx \qquad \text{Bending}$$

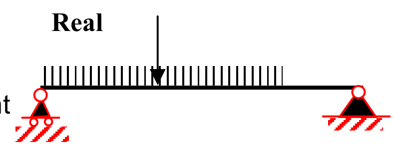
$$+ \int_0^L \frac{f_s \, v\,V}{GA} \, dx \qquad \text{Shear}$$

$$+ \int_0^L \frac{t\,T}{G\,J} \, dx \qquad \text{Torsion}$$

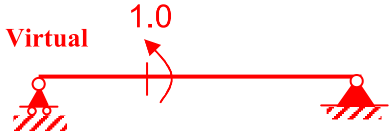
Examples:

1. Determine slope at desired point

Real

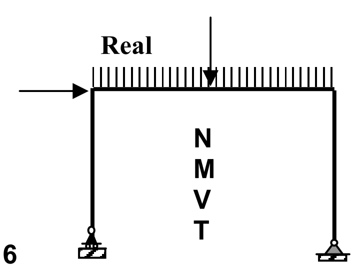


Virtual

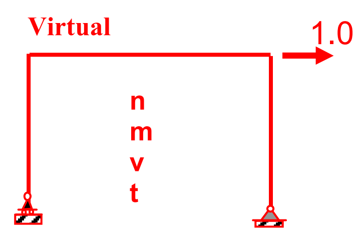


2. Determine horizontal deflection at desired point

Real

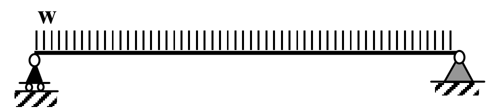


Virtual



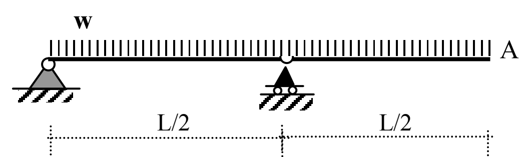
Solved Examples 14-11 to 14-16

Example: Determine the deflection at mid span.

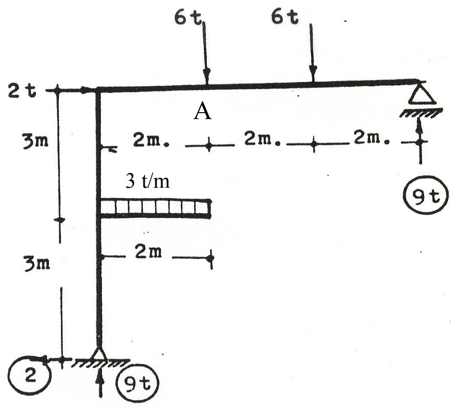


Example: Get deflection at A

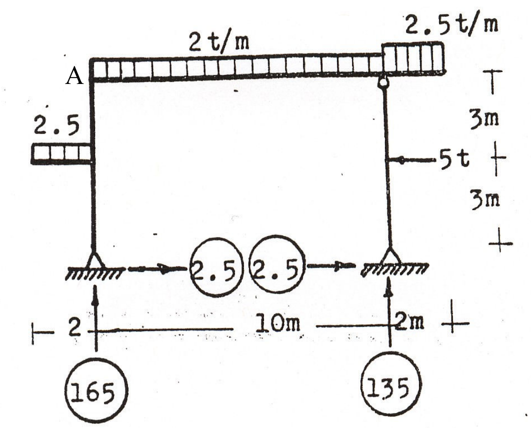
.



Examples on Virtual Work - determine the vertical deflection at point A.



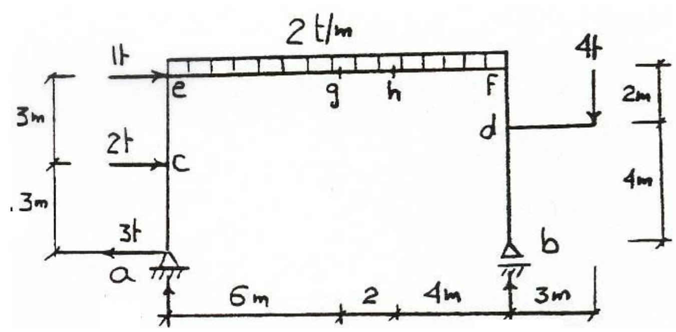
Determine the horizontal deflection at point A.

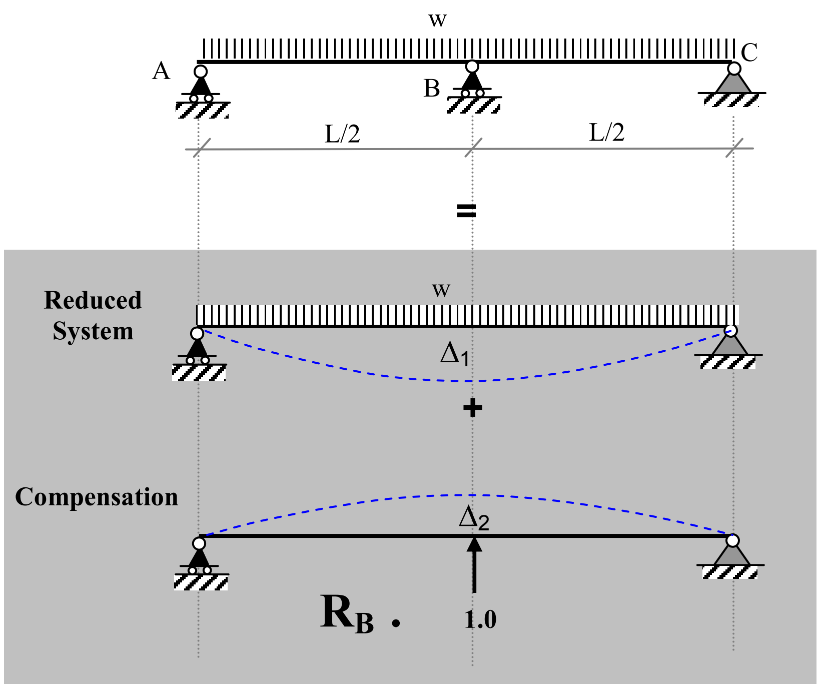


Calculate:

- The horizontal displacement at point b,
- The vertical displacement at point g
- The slope at point f

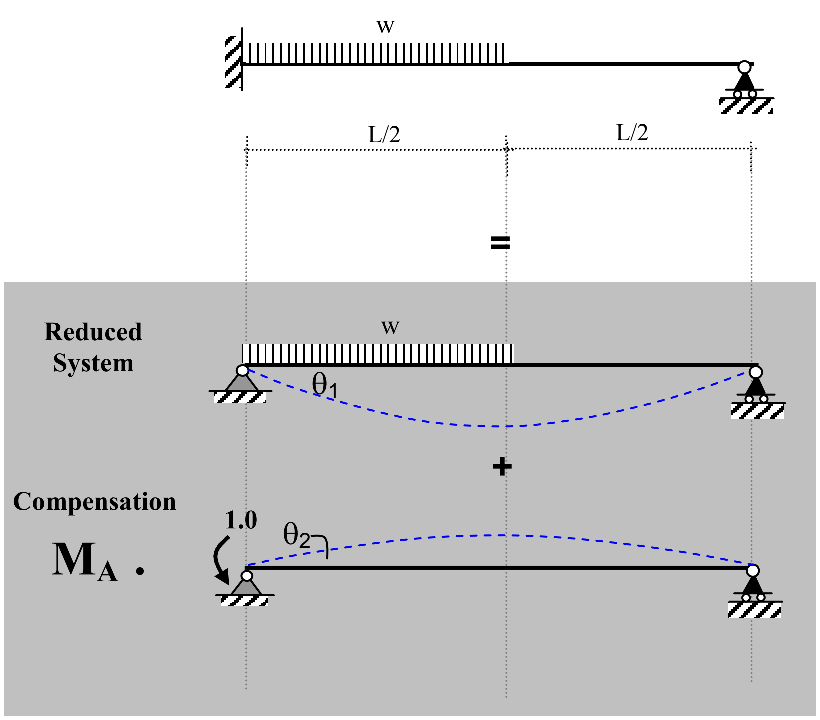
EI = 20,000 m2.t





$$\Delta_1 + R_B\ \Delta_2 = 0$$

Also,



$$\theta_1 + M_A\ \theta_2 = 0$$

Examples:

13. Calculating Deflections Using Castigliano's Theorem

- Put an external load at the position of required deflection: external load (Q) either horizontal or vertical to get horizontal or vertical deflection; or an external moment to get slope.

- Deformation **=** first derivative of the Strain Energy with respect to the applied load.

$$\Delta = dU / dQ \quad \text{, \& substituting } Q = 0$$

$$= \frac{\delta}{\delta Q} \int_0^L \frac{N^2}{2EA}\,dx \qquad = \int_0^L \frac{N}{EA} \frac{\delta N}{\delta Q}\,dx \qquad \text{Axial Load (Trusses)}$$

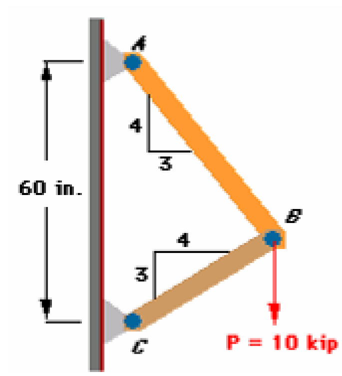
$$= \frac{\delta}{\delta Q} \int_0^L \frac{M^2}{2EI}\,dx \qquad = \int_0^L \frac{M}{EI} \frac{\delta M}{\delta Q}\,dx \qquad \text{Bending Moment}$$

$$= \frac{\delta}{\delta Q} \int_0^L \frac{f_s V^2}{2GA}\,dx \qquad = \int_0^L \frac{f_s V}{GA} \frac{\delta V}{\delta Q}\,dx \qquad \text{Shear}$$

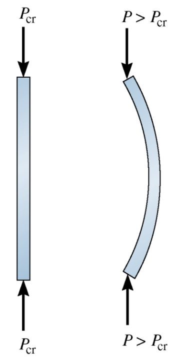
$$= \frac{\delta}{\delta Q} \int_0^L \frac{T^2}{2GJ}\,dx \qquad = \int_0^L \frac{T}{GJ} \frac{\delta T}{\delta Q}\,dx \qquad \text{Torsion}$$

Example:

Determine the horizontal deflection at point B. Cross-section area= 12 in2
E= 30.10^6 psi. AB = 48 in and BC = 36 in.



14. Buckling of Columns



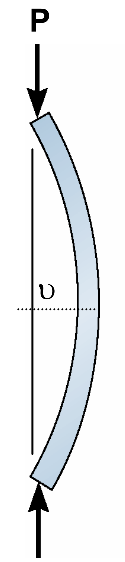
- Slender columns under elastic compression buckle when the load exceeds a critical value.

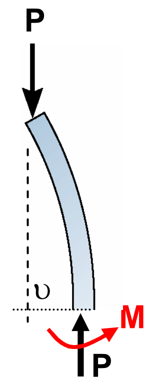
- Buckling causes column instability.

- Short stocky columns do not buckle.

- W need to study the relation between P, Δ, and shape of buckled column.

- Analysis (Euler 1707 – 1783):

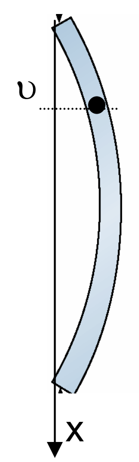




$$M + P.\upsilon = 0$$

Recall, $\dfrac{M}{EI} = \dfrac{d^2\upsilon}{dx^2}$

Then, $$\frac{d^2\upsilon}{dx^2} + \frac{P.\ \upsilon}{EI} = 0$$



Equation of Elastic Curve:

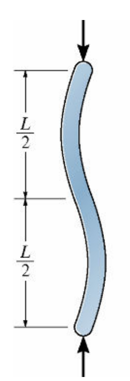
$\upsilon = C1\ \text{Sin}\,[(P/EI)^{0.5}.\ x] \quad + \quad C2\ \text{Cos}\,[(P/EI)^{0.5}.\ x]$

$\upsilon = 0$ at $x = L$ $\qquad\qquad$ $\upsilon = 0$ at $x = 0$

or when, $\text{Sin}\,[(P/EI)^{0.5}.\ L] = 0$ $\qquad\qquad$ $C2 = 0$

or when, $(P/EI)^{0.5}.\ L = \pi, 2\pi, \ldots$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$



$$P_{cr} = \frac{4\pi^2 EI}{L^2}$$

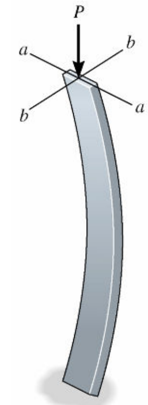
Analysis:

Maximum axial load before buckling:

P/A should be within allowable stresses.

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Smaller of the two directions x & y.

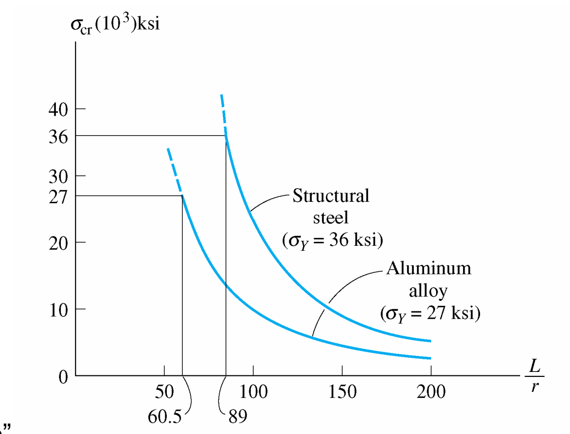


$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{A\,L^2}$$

Put, $r = \sqrt{I / A}$ = radius of gyration

OR

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$



Note that **L/r** is the "Slenderness Ratio" used to classify columns as long, intermediate, or short.

Effect of Column Supports:

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

$KL = L_e$