The Analytic Element Method

ES 661:

Analytical Methods in Hydrogeology

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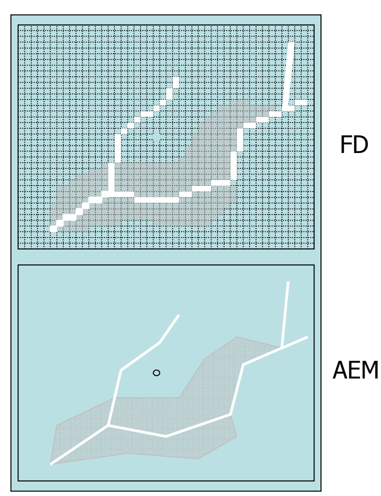
Analytic Element Method

- Alternative numerical method based upon the superposition of simple analytical solutions
  - Grid-independent
    - Discretizes external and internal system boundaries, not entire domain
  - Models limited by amount of detail included, not by spatial extent
  - Exact solution to governing PDE
  - Approximate only in how well BCs are satisfied



Analytic Element Method

- Based upon **superposition** of “element” functions
  - Each element corresponds to a hydrogeologic feature
  - Each element automatically meets governing equations everywhere – exactly!
  - Adjustable (unknown) element coefficients are calculated such that boundary conditions are met
- Solution quality is scale independent
  - No grid/mesh, no worries
- Current major limitations:
  - Heterogeneity: exact, but computationally expensive
  - Transience: computationally expensive and limited
  - 3D Unconfined: the phreatic surface is a tough nut to crack
  - 3D Multilayer (we’re working on it ☺)

Analytic Element Method: History

- Developed by Otto Strack (U. Minnesota), ~ 1980s
  - *Groundwater Mechanics*, 1989
- Popularized by Henk Haitjema (Indiana U.)
  - *Modeling with the Analytic Element Method*, 1995, Academic Press
  - EPA’s WhAEM
- Key developments
  - Surface water interactions (Haitjema and others)
  - Multilayer/Transience (Bakker and Strack and others)
  - Computational improvements (Jankovic, Barnes, Strack, and others)

AEM: Premise

- For any *linear* PDE, we can superimpose multiple individual solutions to obtain one (often very large) solution for the problem at hand
  - Laplace Equation ($\nabla^2\Phi=0$)
  - Poisson Equation ($\nabla^2\Phi=-N$)
  - Helmholtz Equation ($\nabla^2\Phi= \Phi/\lambda^2$)
  - Matrix Helmholtz Equation $\nabla^2\{\Phi\}= [A]\{\Phi\}$
  - Diffusion Equation ($\nabla^2\Phi= 1/\alpha\ \partial\Phi/\partial t$)
  - …

Governing Equations

Governing Equations

- 2D Governing equation for GW Flow:

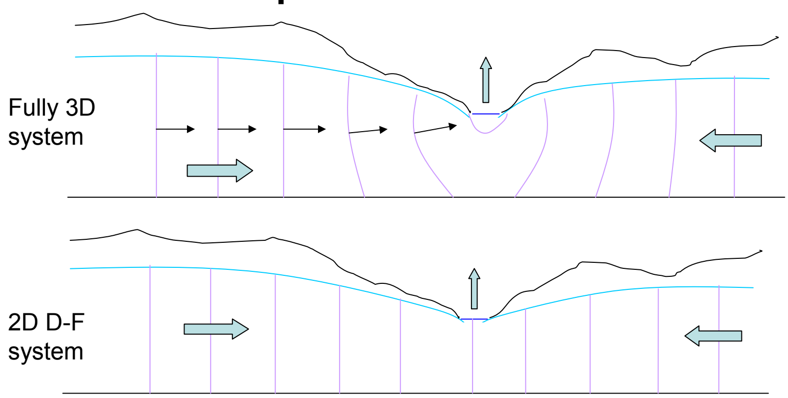
$$\frac{\partial}{\partial x}\underbrace{\left(k_x b\frac{\partial h}{\partial x}\right)}_{Q_x=q_x b} + \frac{\partial}{\partial y}\underbrace{\left(k_y b\frac{\partial h}{\partial y}\right)}_{Q_y=q_y b} = -N + S\frac{\partial h}{\partial t}$$

- Where
  - h = Hydraulic head [L]
  - N = Vertical influx (Rech. or Leakage) [L/T]
  - b = Saturated Thickness [L] (h-B or just H)
  - S = Storage Coeff.[-]

Assumptions

- Dupuit-Forcheimer assumption
  - Required to move from 3D→2D
  - Head may be represented by its average value in the vertical direction / vertical gradients in head are negligible (dh/dx≈0)
  - Resistance to flow is negligible in the vertical direction (i.e., $k_z \approx \infty$)
  - q_z calculated from mass balance in vertical, rather than by using Darcy's law (Strack, 1984)
  - Appropriate for systems with much greater horizontal than vertical extent

Dupuit-Forcheimer



Average heads in vertical direction are the same

Vertical distribution of heads is lost

Water balance is still conserved! (in fact, Q_x/Q_y are still exact)

Governing Equations

- By assuming
  - isotropy ($k=k_x=k_y$)
  - homogeneity (k, H, and B are piecewise constant)
- we can define a discharge potential:

$$\Phi = kHh - \frac{1}{2}kH^2 \qquad (b \geq H) \quad \text{Confined}$$

$$\Phi = \frac{1}{2}kh^2 \qquad (b < H) \quad \text{Unconfined}$$

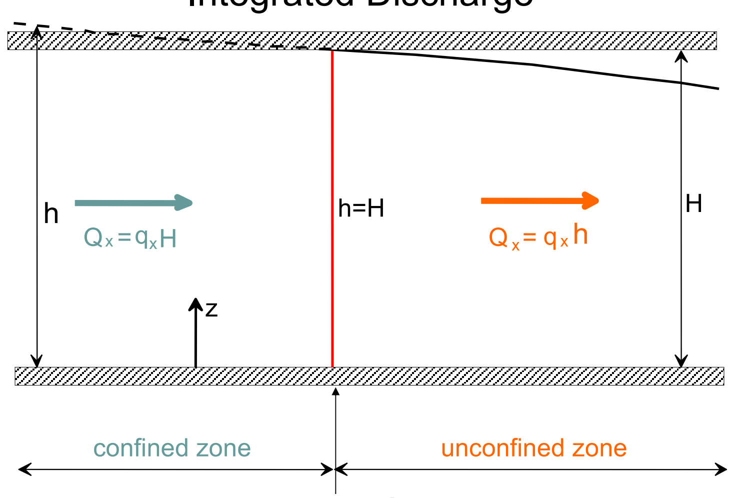
Discharge Potential

- The discharge potential is the antiderivative of the integrated discharge

$$Q_x = -\frac{\partial \Phi}{\partial x} \quad Q_y = -\frac{\partial \Phi}{\partial y}$$

- i.e., if we know Φ (and k,H,B) we can backcalculate h, Qx, Qy

Integrated Discharge



$\Phi = \frac{1}{2}kh^2$ > $\Phi = \frac{1}{2}kH^2$ > $\Phi = kHh - \frac{1}{2}kH^2$

Simplifying things

- Using the discharge potential, we can rewrite our governing equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \nabla^2 \Phi = -N + \cancel{S\frac{\partial h}{\partial t}}$$

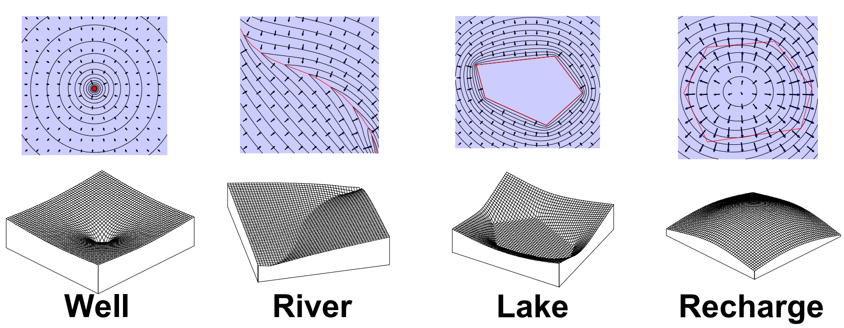
- Focus on steady-state (for now)

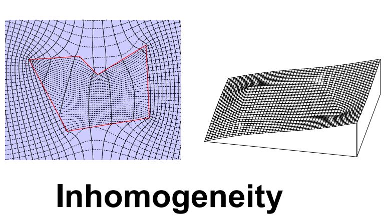
Analytic Elements

Analytic Elements

- Because our governing equation is linear, we may superimpose ANY particular analytical solutions to get at a global solution
- These particular solutions are “elements”, which generally correspond to hydrogeologic features
  - Pumping wells
  - Rivers/Lakes/Streams
  - Inhomogeneities in K, B, H
- Each element satisfies the governing equation by design and has adjustable coefficients which can be used to satisfy boundary conditions along its border
- Calculating the appropriate coefficient values is where the numerical part comes in

Standard Analytic Elements





Elementary solutions superimposed to obtain complete description of flow system...

Superposition Mathematics

$$\frac{\partial^2(\Phi_1+\Phi_2)}{\partial x^2}+\frac{\partial^2(\Phi_1+\Phi_2)}{\partial y^2}=$$

$$\left(\frac{\partial^2\Phi_1}{\partial x^2}+\frac{\partial^2\Phi_1}{\partial y^2}\right)+\left(\frac{\partial^2\Phi_2}{\partial x^2}+\frac{\partial^2\Phi_2}{\partial y^2}\right)$$

Laplacian of a sum of potentials equals the sum of Laplacians of individual potentials

Therefore, we can write our global solution as:

$$\Phi(x,y)=\Phi_{well}(x,y)+\Phi_{river}(x,y)+\Phi_{lake}+...+C$$

(Assuming all of the $\Phi(x,y)$ functions satisfy the Laplace equation)

Complex Potential

- Most of our 2D SS analytic elements are actually expressed in terms of a complex potential, $\Omega(z)$:

$$\Omega(z)=\Phi(z)+i\Psi(z)$$

Where z=x+iy (i=√-1)

- This is because ANY infinitely differentiable (a.k.a. analytic) complex function instantly has real and imaginary parts that both satisfy the Laplace equation, by definition- if we start with *any* analytic function, we are halfway to our goal
  - These simple functions are our "building blocks"

$$a \quad a\ln(z) \quad \sum_{n=0}^{N} a_n z^n \quad \sum_{n=0}^{N} a_n z^{-n} \quad \sum_{n=0}^{N} a_n e^{nz} \cdots$$

Stream Function, Ψ

- Imaginary part of complex potential, Ω
- Defined only if N=0 (no recharge, no leakage)
- Constant along streamlines
- Difference in Ψ between streamlines equals flow between streamlines

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$$

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0$$

Complex Discharge, W

- Just an expression for the discharge in terms of complex functions

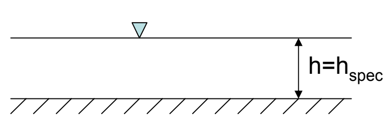
$$W = -\frac{\partial\Omega}{\partial z} = Q_x - iQ_y$$

Simplest Elements

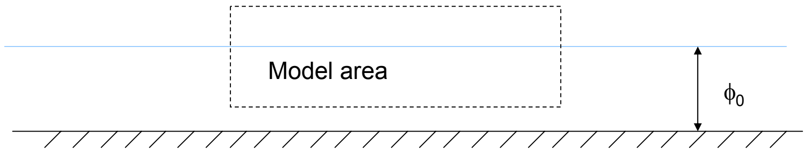
- The Global Constant, C
- Uniform Flow
- Wells
- Linesinks
- Line doublets

The global constant, Φ_0=C

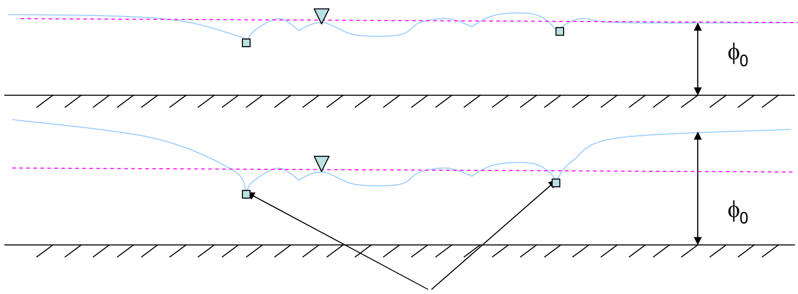
- The "baseline" of our model
  - If there are no forcing functions in our model (i.e., wells, rivers, etc.), it is the potential everywhere in the domain.
- It is usually calculated by specifying the head at a distant point (the "reference point")
- Mathematical necessity- essentially specifies the boundary condition at infinity (AEM works with an infinite model domain)



Global constant, Φ_0



No effect in a well bounded model (i.e., modeled domain is not infinite):



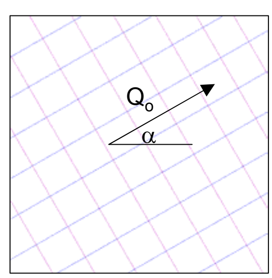
Head-specified model boundaries (e.g., rivers) extract more to compensate

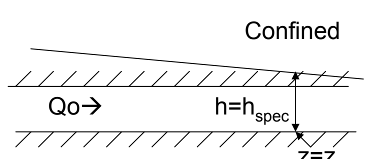
Uniform Flow

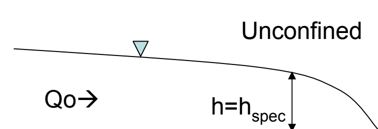
- Used to represent influence of distant features not included in model

$$\Omega_{uf}(z) = -Q_0(z - z_{ref})e^{i\alpha}$$

$$\begin{aligned} W_{uf}(z) &= Q_0 e^{i\alpha} \\ &= \underbrace{(Q_0 \cos\alpha)}_{Q_x} - i\underbrace{(-Q_0 \sin\alpha)}_{Q_y} \end{aligned}$$







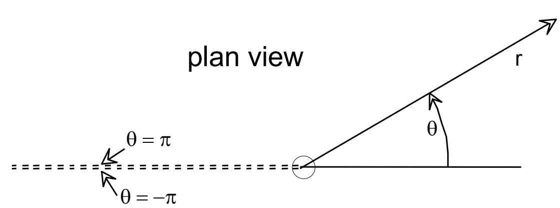
Point Sink

- The steady-state influence of extraction at a point
- a.k.a. the Thiem solution for a well

$$\Omega_{wl}(z) = \frac{Q_w}{2\pi} ln(z - z_w)$$

- The basis for many of our standard elements – the function $\ln(|z|)/2\pi$ is actually the Green's function for the Laplace equation

Complex Potential Due to a Well



$$\Omega_{wl}(z) = \frac{Q_w}{2\pi} ln(z - z_w)$$

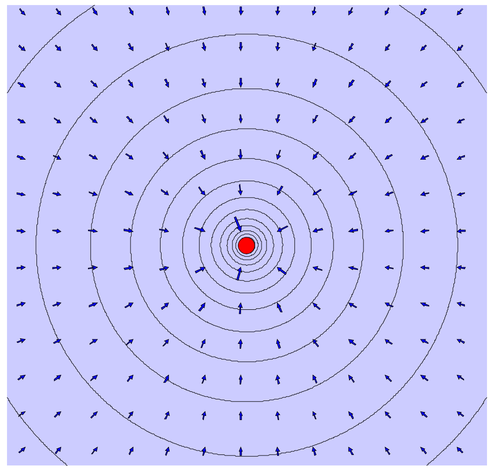
$$\Phi_{wl}(z) = \frac{Q_w}{2\pi} ln(r)$$

$$\Psi_{wl}(z) = \frac{Q_w}{2\pi} \theta$$

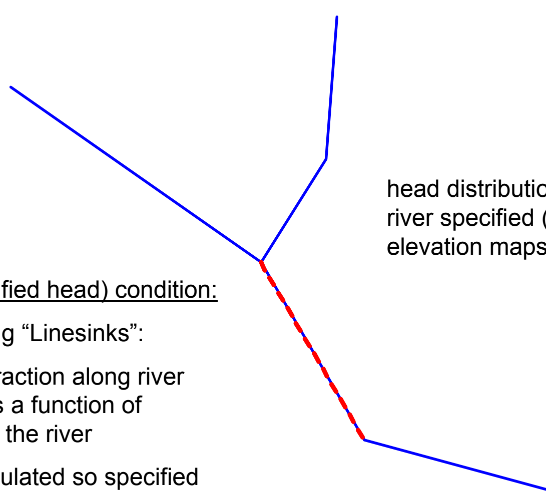
$$W_{wl}(z) = \frac{Q_w}{2\pi(z - z_w)}$$

Q_w	=Extraction Rate [L^3/T]
$z = x+iy$	=Location where Ω is evaluated
$z_w = x_w + iy_w$	=Location of well
$r = \lvert z - z_w \rvert$	$= \sqrt{[(x-x_w)^2 + (y-y_w)^2]}$
$\theta = \arg(z - z_w)$	$= \arctan(y-y_w/x-x_w)$

Potential Due to a Well



Element: River



head distributions along the river specified (using digital elevation maps)

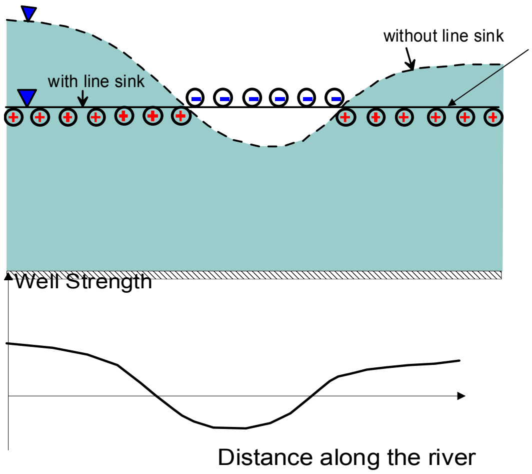
<u>Dirichlet (specified head) condition:</u>

Simulated using “Linesinks”:

distributed extraction along river represented as a function of distance along the river

Extraction calculated so specified head is obtained

Linesink



• Well strength is represented using continuous functions with unknown coefficients

• Coefficients are computed from specified head distribution

• Integrated distribution of well strength gives baseflow to the river

Linesink

- N evenly spaced wells of pumping rate Qn may be superimposed to get:

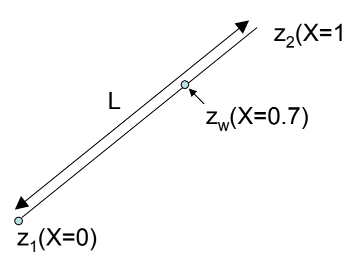
$$\Omega(z) = \sum_{n=1}^{N} \frac{Q_n}{2\pi} ln(z - z_w\left(\frac{n}{N}L\right)))$$

$$z_w(X) = z_1 + \frac{X}{L}(z_2 - z_1)$$

- Taking the limit as N→∞,

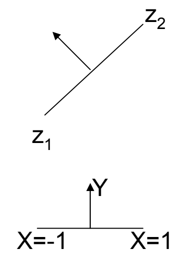
$$\Omega(z) = \int_0^L \frac{\mu(X)}{2\pi} ln(z - z_w(X))dX$$

This integral can be evaluated analytically if the distributed pumping rate, μ(X), is a polynomial



Linesink: Uniform Strength

- If $\mu(X)=\sigma$ (constant), then we get a basic linesink:



$$\Omega(z) = \frac{\sigma L}{4\pi}\left[(Z+1)\ln(Z+1) - (Z-1)\ln(Z-1)\right]$$

Where $$Z = X + iY = \frac{z - 0.5(\frac{1}{z} + \frac{2}{z})}{0.5(\frac{1}{z} + \frac{2}{z})}$$

- σ can be calculated to meet a specified head at one point along the line (collocation) or calculated to meet a specified in the best manner possible at many points (least squares)

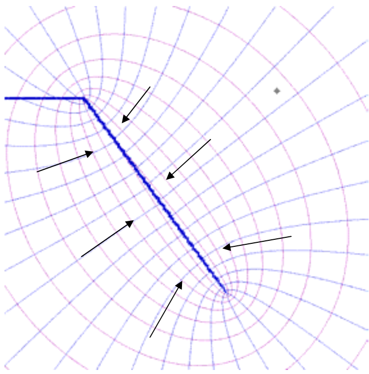
Linesink: Arbitrary Strength

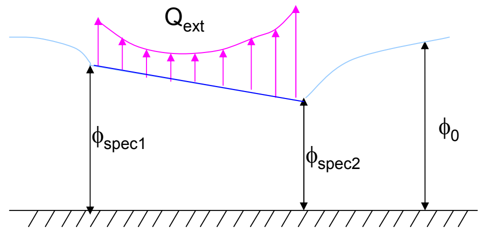
- If $\mu(X)$ is an arbitrary function (usually a polynomial), then we get a high-order linesink:

$$\Omega_{ls}(Z) = \frac{1}{2\pi}\left(\mu(Z) ln\frac{Z-1}{Z+1} + q(Z)\right) - \frac{\mu(1)}{2\pi} ln(Z-1) + \frac{\mu(-1)}{2\pi} ln(Z+1)$$

- Here, q(Z) is used to ensure that the influence of the linesink dies off as 1/r in the distance (for numerical stability)- it is directly calculated from $\mu(X)$
- We can calculate the coefficients of $\mu(X)$ to best meet our desired boundary condition

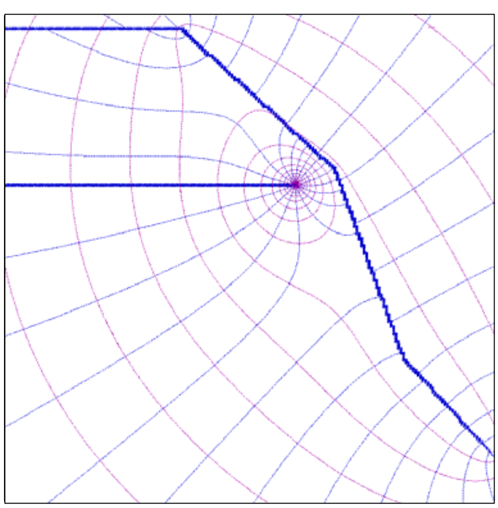
Rivers: Head specified

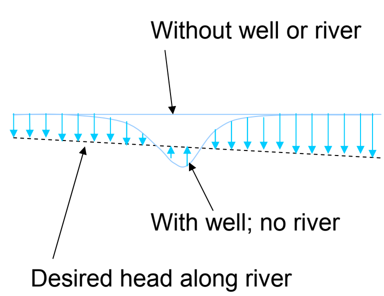




Extracts enough water along boundary to meet head specified conditions

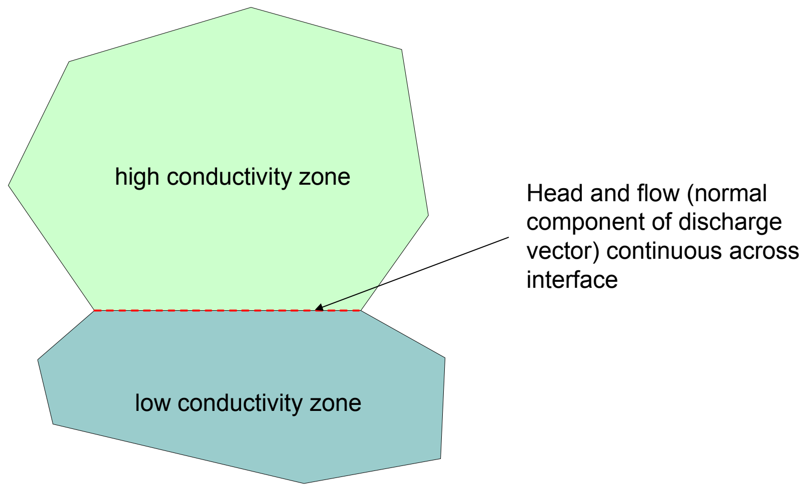
Example: Head-specified element



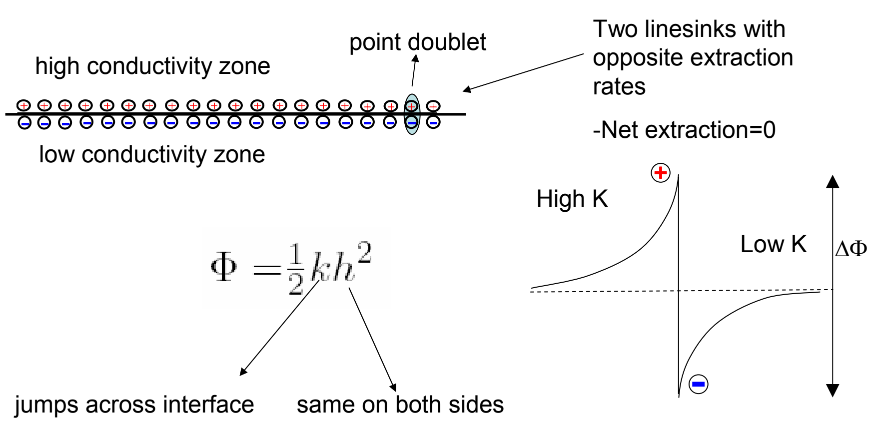


River adds/removes enough water along its border to meet specified head boundary conditions

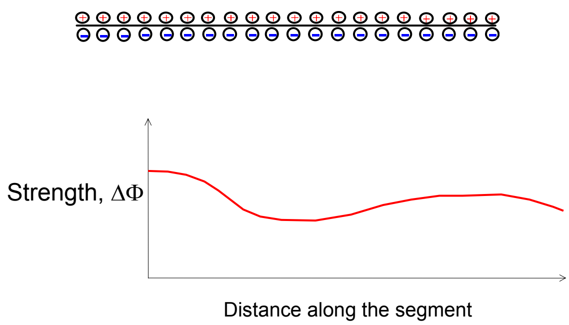
Boundary Condition: Change in Conductivity



Analytic Element: Line Doublet

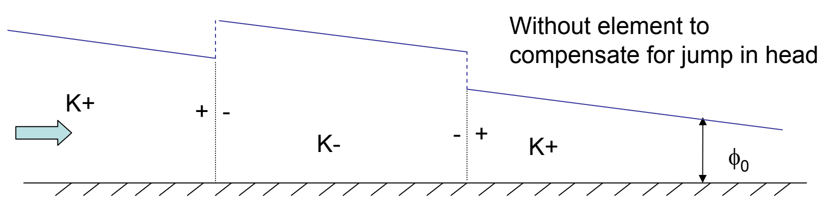


Analytic Element: Line Doublet

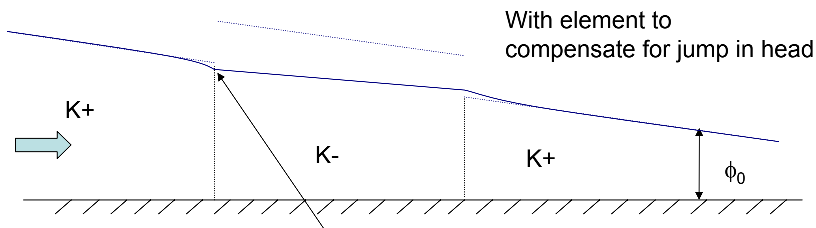


- Doublet strength is represented using continuous functions with unknown coefficients
- Coefficients are computed by enforcing head continuity
- Total amount of water added to (or extracted from) the aquifer is **always** zero

Inhomogeneities: Higher K zone

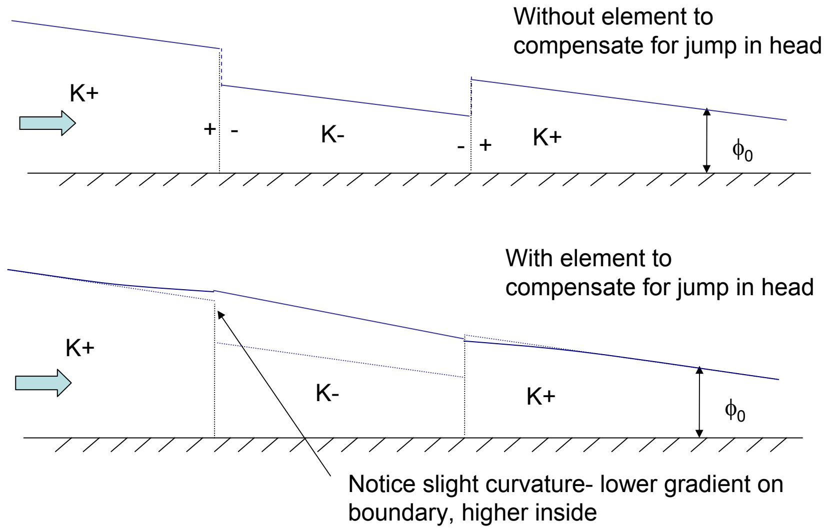


Changing K creates discontinuity in head
(Φ from other elements is continuous by definition)

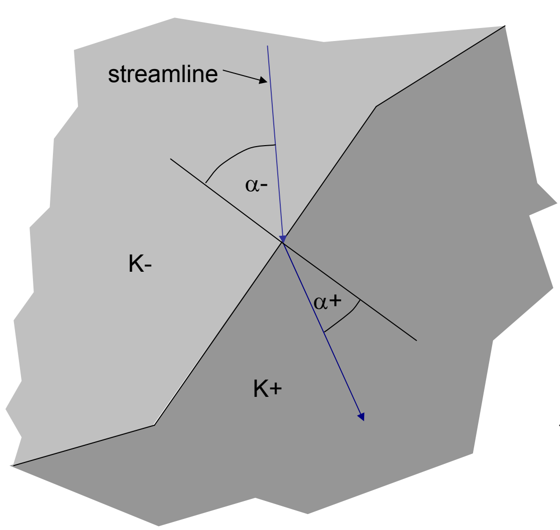


Notice slight curvature- higher gradient on boundary, lower gradient inside

Inhomogeneities: Lower K zone



Law of Refraction



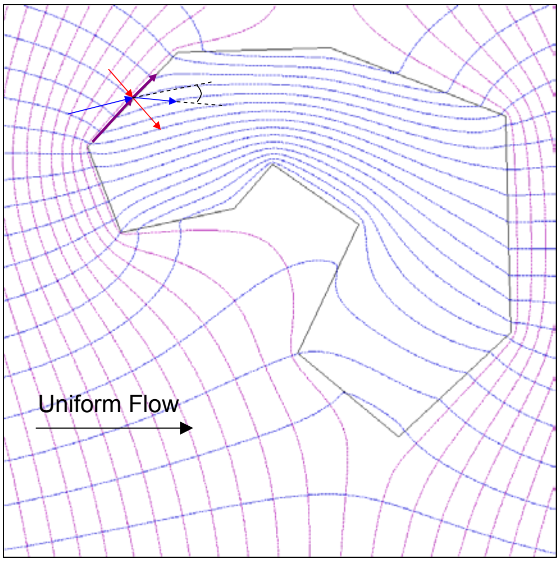
Normal component of flux continuous across change in conductivity

Tangential component changes

Change in ratio Q_n/Q_t (and thus streamline angle) proportional to change in K

$$\frac{\tan\alpha^+}{K^+} = \frac{\tan\alpha^-}{K^-}$$

Example: Inhomogeneity



Highly conductive

Kin/Kout=10

Kin/Kout=2

Kin=Kout

Kin/Kout=0.5

Kin/Kout=0.1

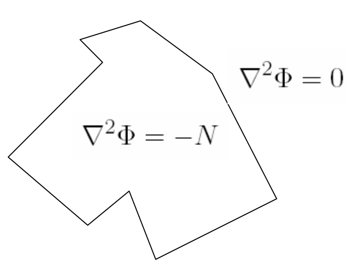
Impermeable

Inhomogeneity bends streamlines along its border in order to meet law of refraction

This is the same as trying to meet a jump in potential to preserve continuity of head

Area Sinks

- Satisfies Poisson equation inside (i.e., N≠0) polygon or circle, and Laplace Equation outside

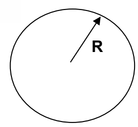


Circular Area Sink

- Simplest Case – radial symmetry

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} = -N \qquad \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} = 0$$

- Basic solution:



inside $\Phi = -\frac{1}{4}Nr^2 + C$

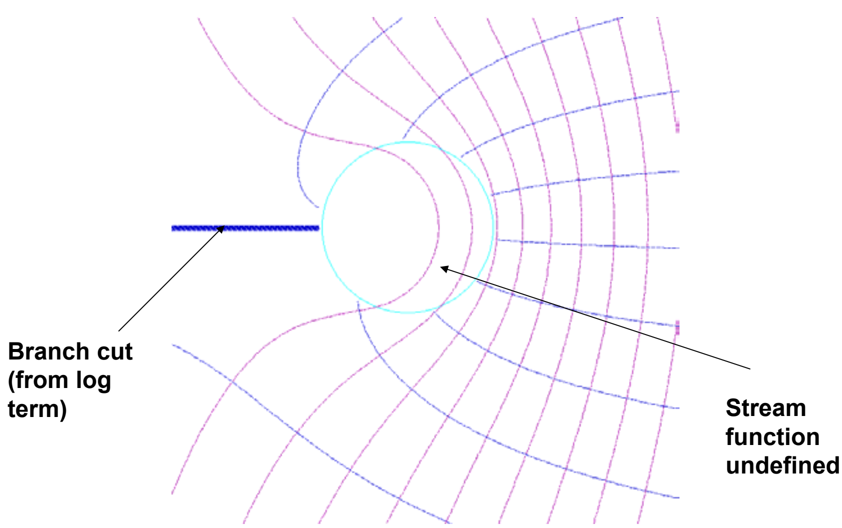
outside $\Phi = A\ln(r) + B$

3 unknowns, 2 eqns :
continuity of potential/head at r=R
Conservation of Net flux ($2\pi A = -N\pi R^2$)

$\Phi = -\frac{1}{4}N[r^2 - R^2]$ **+D** $\qquad \Phi = -\frac{1}{2}NR^2\ln(r/R)$ **+D**

D is “folded into” global constant

Circular Area Sink in Uniform Flow



AEM: Solution Method

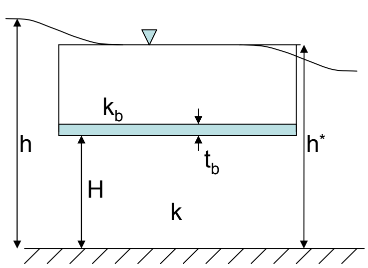
- All of the AEM elements have adjustable coefficients
- For each coefficient we can write an equation to
  - Meet a boundary condition at a point or
  - Meet a boundary condition in the best manner at a set of points (least-squares)
- This results in a **fully-populated** system of equations
  - Potential at any point is determined by the sum of all potential functions
  - Each equation includes all unknown coefficients

AEM Software

- Freeware
  - Visual Bluebird (soon to be Visual AEM)
    - http://www.groundwater.buffalo.edu/software/
  - WhAEM (US EPA)
  - TimML (UGA)
- $
  - MLAEM/SLAEM (Otto Strack)
  - GFlow
  - TwoDAN

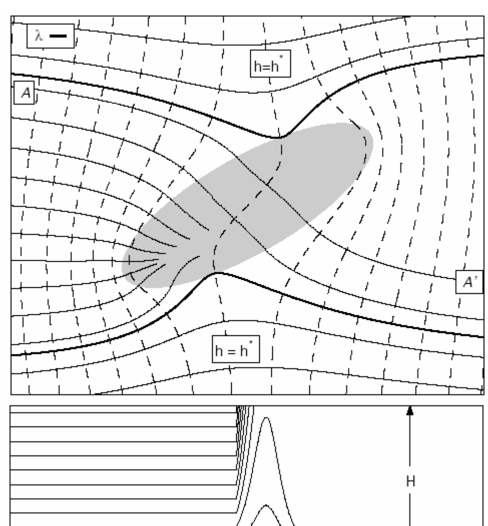
Advanced AEM:
Hot Research Topics

AEM for Resistance Elements



$c = k_b / t_b$

$\lambda = \sqrt{kHc}$



AEM for 3D flow

- The Laplace equation is still valid in 3D, except in terms of a specific discharge potential (a.k.a. velocity potential)

$$\frac{\partial}{\partial x}\left(k\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial h}{\partial z}\right)=0$$

- Φ=kh→

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

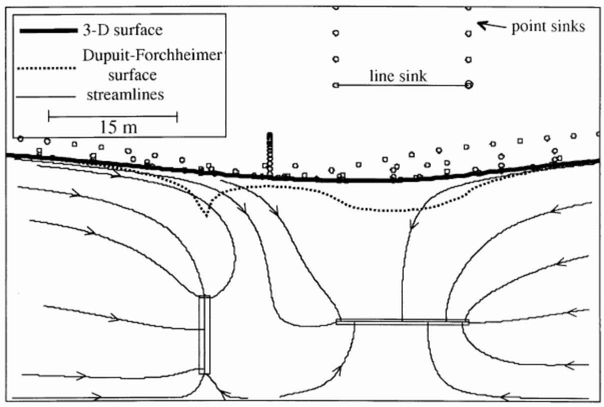
- A major problem is that our system is not infinite in 3 dimensions
  - Phreatic surface
  - Confining layer

AEM for 3D Flow

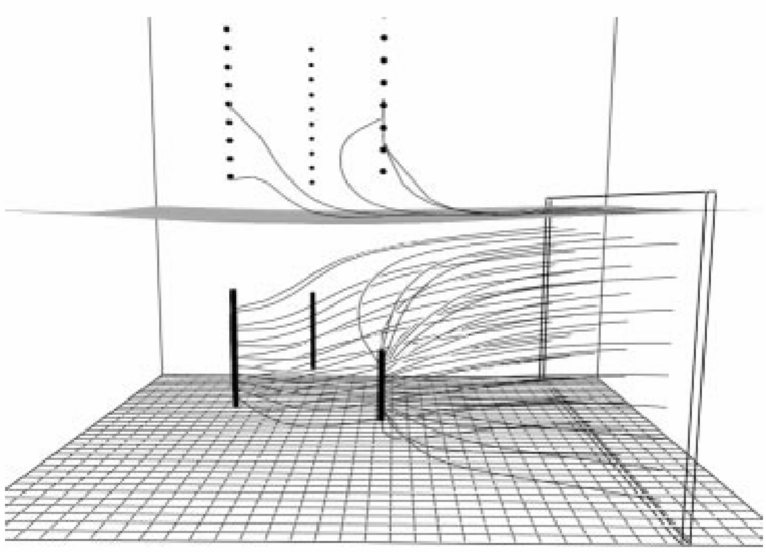
- We have point sinks, line sinks, and ellipsoidal “doublets” (inhomogeneities), but we don’t have the solution for an arbitrary panel (i.e., a 3D triangular doublet/sink)
- Limits the applicability to unconfined systems

AEM for 3D Flow

- Phreatic surface generated using “image” sinks

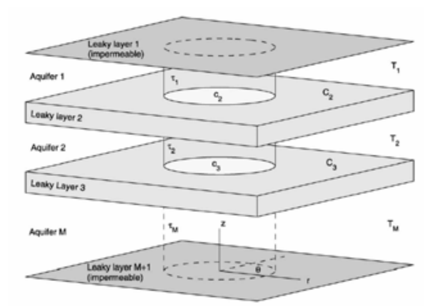


AEM for 3D Flow



AEM for Multilayer Aquifers

- Most work done by Mark Bakker at UGA
- Based upon theories proposed by Hemker (1984)
- Bakker & Strack (Journal of Hydrology 2003)



AEM for Multilayer Aquifers

- Governing Matrix Differential equation (Helmholtz) – D-F Assumption in each layer

$$\Phi_m = k_m H_m h_m$$

$$\nabla^2 \Phi_n = \sum_{m=1}^{N} A_{n,m} \Phi_m \qquad n = 1, \ldots, N,$$

$$A_{n,n} = \frac{1}{c_n T_n} + \frac{1}{c_{n+1} T_n},$$

$$A_{n,n-1} = -1/(c_n T_{n-1}),$$

$$A_{n,n+1} = -1/(c_{n+1} T_{n+1}).$$

A is tridiagonal matrix which handles the “communication” between layers

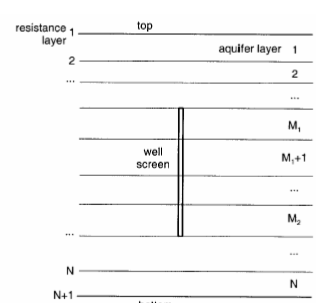


Figure 1. Schematic cross section of the aquifer.

AEM for Multilayer Aquifers

- Solved using eigenmethods, general solution given as:

$$\vec{\Phi} = \Phi_L \vec{\tau} + \sum_{m=1}^{M-1} \Phi_m \vec{v}_m \qquad \nabla^2 \Phi_L = 0 \qquad \nabla^2 \Phi_m = \Phi_m / \lambda_m^2$$

- Where tau is the transmissivity vector

$$\vec{\tau} = \frac{\overrightarrow{kH}}{T}$$

T is the comprehensive transmissivity

- And ν_m are the eigenvectors of A

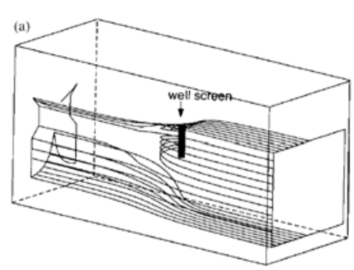
AEM for Multilayer Aquifers

- Solution for Well (Bakker, 2001)

$$\vec{\Phi} = \boxed{\frac{Q}{2\pi}} \ln(r) \vec{\tau} + \sum_{m=1}^{M-1} \boxed{\frac{A_m}{2\pi} \mathrm{K}_0(r/\lambda_m)} \vec{v}_m$$

Standard 2D SS solution

Redistributes head between layers

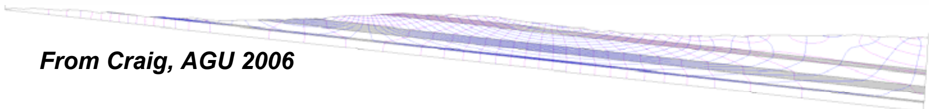


Where A_m are obtained from the following system of equations

$$\sum_{m=1}^{M-1} A_m v_{p,m} = Q\tau_p, \quad p = 1, \ldots, M; \quad p \neq P$$

- Most solutions expressed in terms of Bessel and Mathieu functions
- Available from TimML webpage

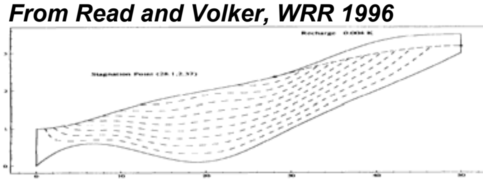
AEM for 3D Multilayer Aquifers

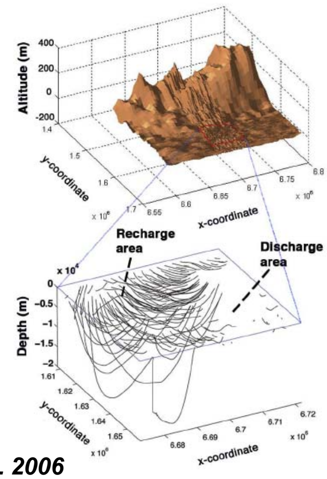


From Craig, AGU 2006

- A different approach: Series solution methods on finite domains

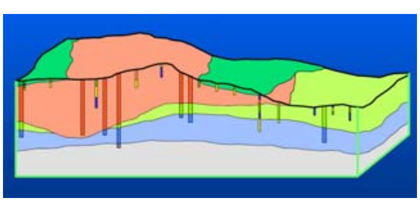
From Read and Volker, WRR 1996

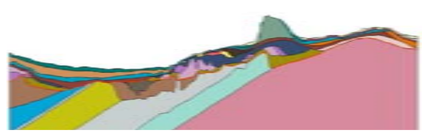


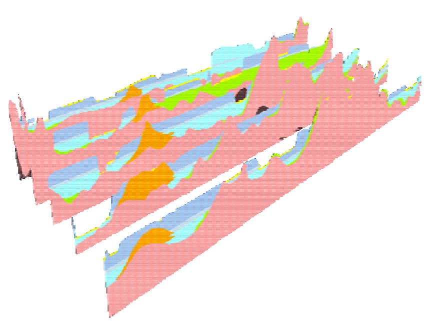


From Wörman et al., GRL 2006

AEM for 3D Multilayer aquifers







AEM for Transient Systems

- Introduced by Furman and Neuman (2004)
- Governing Equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = S\frac{\partial h}{\partial t} \quad \rightarrow \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{1}{\alpha}\frac{\partial \Phi}{\partial t}$$

- Where

$$\Phi = kh \qquad \alpha = \frac{k}{S}$$

- This can be solved in Laplace Transformed domain as the Helmholtz eqn. and *numerically inverted*

AEM for Transient Aquifer Systems

- The LT-AEM currently has a small (but growing) library of elements
  - Wells
  - Circular and Elliptical elements
  - Linesinks (from degenerate ellipses)
    - Kuhlman, 2006 (personal comm.)
- Limited to confined conditions

AEM for Smoothly Heterogeneous Aquifers?

- ln k represented by radial basis functions
- If $Y' = \ln \kappa$, where $k = \kappa^2 \cdot \bar{k}$:

$$\nabla^2 \Phi = \left[\frac{\partial Y'}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{\partial Y'}{\partial y} \frac{\partial \Phi}{\partial y} \right]$$

- Or (via Bers-Vekua theory):

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial Y'}{\partial \bar{z}} \bar{w}$$

$$w = \kappa^2 \Phi + i \kappa^{-2} \Psi$$