Isospectral Systems

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Abstract

Two vibrating systems are said to be isospectral if they have the same natural frequencies. The paper examines how families of isospectral systems may be constructed. The examples which are considered include rods, beams and simple finite element models.

1. INTRODUCTION

The study of isospectral systems is an integral part of the theory of Inverse Problems in Vibration; families of isospectral systems appear as solutions of inverse problems in which there is unsufficient spectral data to yield a unique vibrating system.

This paper concerns the undamped, free, infinitesimal vibration of a discrete elastic system $\mathcal{S}$ about an equilibrium configuration. The governing equation for such a system has the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{1}$$

where $\mathbf{M}, \mathbf{K}$ are the mass and stiffness matrices of $\mathcal{S}$. The natural frequencies $(\omega_i)_1^n$ of $\mathcal{S}$ are related to the eigenvalues $(\lambda_i)_1^n$ of

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{q} = \mathbf{0} \tag{2}$$

by $\lambda_i = \omega_i^2$. The set $(\lambda_i)_1^n$ is called the *spectrum* of $\mathcal{S}$, and is denoted by $\sigma(\mathbf{M}, \mathbf{K})$. Two systems $\mathcal{S}$ and $\mathcal{S}'$ are said to be *isospectral* if

$$\sigma(\mathbf{M}', \mathbf{K}') = \sigma(\mathbf{M}, \mathbf{K}). \tag{3}$$

In general, the most that can be stated about $\mathbf{M}$ and $\mathbf{K}$ is that they are symmetric; $\mathbf{M}$ is positive definite (PD); $\mathbf{K}$ is positive semi-definite (PSD), and PD if the system is anchored. These conditions ensure that the spectrum $\sigma(\mathbf{M}, \mathbf{K})$ is non-negative, positive if $\mathcal{S}$ is anchored.

Since $\mathbf{M}$ is PD, it may be factorized in the form $\mathbf{M} = \mathbf{L}\mathbf{L}^T$ where $\mathbf{L}$ is lower triangular, and (2) may be reduced to standard form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{0}, \tag{4}$$

where $\mathbf{A} = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$ is PSD, and $\mathbf{L}^T\mathbf{q} = \mathbf{u}$. The matrix $\mathbf{A}$ is called the *mass-reduced stiffness matrix* (MRK) of $\mathcal{S}$; the spectrum of $\mathbf{A}$ is denoted $\sigma(\mathbf{A})$. It is well known that two symmetric matrices $\mathbf{A}, \mathbf{A}'$ are isospectral, $\sigma(\mathbf{A}') = \sigma(\mathbf{A})$, iff (if and only if)

$$\mathbf{A}' = \mathbf{Q}^T\mathbf{A}\mathbf{Q}, \tag{5}$$

where $\mathbf{Q}$ is an orthogonal matrix, i.e., $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. This means that, at the level of MRK's, the search for isospectral systems is straightforward: two systems $\mathcal{S}$ and $\mathcal{S}'$ are isospectral iff their MRK's are linked by (5) for some orthogonal matrix $\mathbf{Q}$. However, this does not end the search for isospectral systems at the physical (mechanical) level. There the search is for matrices $\mathbf{M}, \mathbf{K}$ which have a *specified form* defined by the connectivity of $\mathcal{S}$. This pattern is mirrored in the pattern of zero and non-zero elements of $\mathbf{K}$ and/or $\mathbf{M}$. There may also be specified *sign patterns* for the elements of $\mathbf{K}$ and/or $\mathbf{M}$. If $\mathbf{K}, \mathbf{M}$ have such specified forms, $\mathbf{A}$ is constructed as $\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$, and $\mathbf{Q}$ is an arbitrary orthogonal matrix, then there is no guarantee that $\mathbf{A}'$, constructed as in (5), may be written as $\mathbf{A}' = (\mathbf{L}')^{-1}\mathbf{K}'(\mathbf{L}')^{-T}$, where $\mathbf{K}'$ and $\mathbf{M}' = \mathbf{L}'\mathbf{L}'^T$ also have the specified forms. This is the crux of the problem: find those orthogonal matrices $\mathbf{Q}$ which guarantee such an invariance of form. We consider a number of specified forms for $\mathbf{M}, \mathbf{K}$, starting with a simple example and proceeding to more demanding ones.

2. IN-LINE MASSES AND SPRINGS

This system, shown in Figure 1(a), has been studied exhaustively; for references, see Gladwell (1986). The system matrices are

$$\mathbf{K} = \mathbf{E}\hat{\mathbf{K}}\mathbf{E}^T, \quad \mathbf{M} = diag(m_1, m_2, \ldots, m_n), \tag{6}$$

where $\hat{\mathbf{K}} = diag(k_1, k_2, \ldots, k_n)$, and $\mathbf{E}$ is the difference matrix with inverse E^{-1} given by

$$\mathbf{E} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{bmatrix}, \quad \mathbf{E}^{-1} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ & 1 & \ldots & 1 \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{7}$$

The MRK is

$$\mathbf{A} = \mathbf{D}^{-1}\mathbf{E}\hat{\mathbf{K}}\mathbf{E}^T\mathbf{D}^{-1}, \tag{8}$$

where $\mathbf{M} = \mathbf{D}^2$. $\mathbf{A}$ is a PD tridiagonal matrix with negative co-diagonal, a so-called *Jacobi* matrix. Gladwell (1986) shows that once $\mathbf{A}$ is known, $\mathbf{K}$ may be found so that it has the characteristic property of a stiffness matrix,

$$\mathbf{K}\{1, 1, \ldots, 1\} = \{k_1, 0, \ldots, 0\}, \tag{9}$$

by solving

$$\mathbf{A}\{d_1, d_2, \ldots, d_n\} = \{k_1/d_1, 0, \ldots, 0\}. \tag{10}$$

It is known that $\mathbf{K}, \mathbf{M}$ may be constructed uniquely, apart from a scale factor, from either of two equivalent sets of data:

i) $\sigma \equiv \sigma(\mathbf{A}) = (\lambda_i)_1^n; \quad \sigma_n = \sigma_n(\mathbf{A}) = (\mu_i)_1^{n-1}$

ii) $\sigma \equiv \sigma(\mathbf{A}) = (\lambda_i)_1^n; \quad P = (u_n^{(1)}, u_n^{(2)}, \ldots, u_n^{(n)})$

Here $\sigma_n(\mathbf{A})$ means the spectrum of the matrix obtained by deleting the nth row and column of $\mathbf{A}$; this is the spectrum of the system of Figure 1(b). The two spectra σ, σ_n must *interlace*, i.e.,

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n. \tag{11}$$

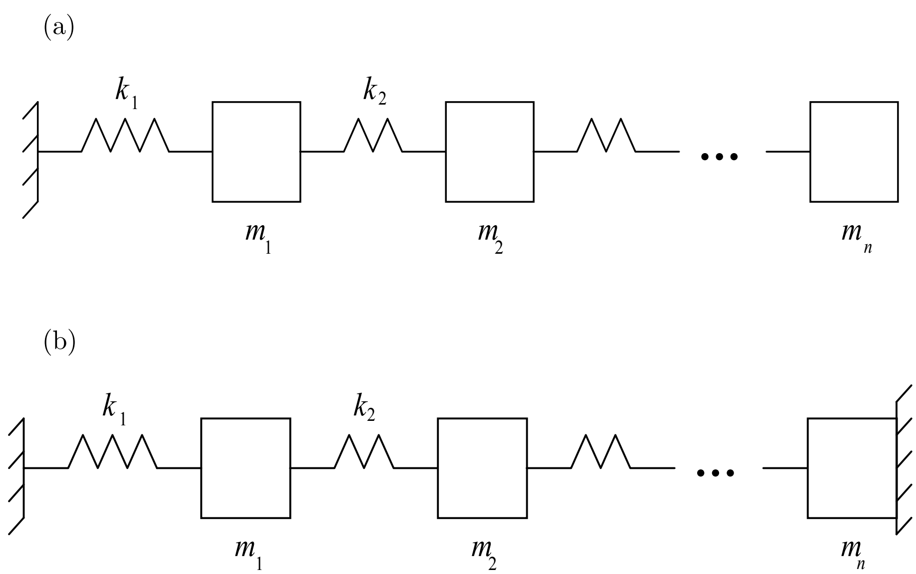


Figure 1 - A spring-mass system with the right hand end (a) free (b) fixed.

In ii), $u_n^{(i)}$ denotes the nth element of the ith normalized eigenvector of $\mathbf{A}$, i.e., scaled so that

$$\sum_{i=1}^{n} (u_n^{(i)})^2 = 1. \tag{12}$$

It is known that $u_n^{(i)} \neq 0$. This means that there is a 1-1 correspondence between each Jacobi matrix in the isospectral family $\sigma(\mathbf{A}) = \sigma$, and the point P in the strictly positive orthant of the hypersphere $\mathcal{S}$ given by (12).

Corresponding to i), ii), there are two ways of finding other Jacobi matrices $\mathbf{A}'$ such that $\sigma(\mathbf{A}') = \sigma(\mathbf{A}) = \sigma$:

i) construct $\mathbf{A}'$ from σ and $\sigma_n(\mathbf{A}') = (\mu_i')_1^{n-1}$, where σ and $\sigma_n(\mathbf{A}')$ interlace as in (11);

ii) construct $\mathbf{A}'$ from σ and P' on $\mathcal{S}$. Going from $\mathbf{A}$ to $\mathbf{A}'$ may then be viewed as a displacement along the arc PP' on $\mathcal{S}$.

There is another way to construct $\mathbf{A}'$. Gladwell (1995) showed that if μ is not an eigenvalue of $\mathbf{A}$; if $\mathbf{A} - \mu\mathbf{I}$, which is non-singular, is factorized as

$$\mathbf{A} - \mu\mathbf{I} = \mathbf{QR}, \tag{13}$$

where $\mathbf{Q}$ is orthogonal, and $\mathbf{R}$ is upper triangular with *positive* diagonal (this factorization is unique), then $\mathbf{A}'$ defined by

$$\mathbf{A}' - \mu\mathbf{I} = \mathbf{RQ} \tag{14}$$

is a Jacobi matrix isospectral to $\mathbf{A}$. Clearly,

$$\mathbf{A}' = \mu\mathbf{I} + \mathbf{RQ} = \mathbf{Q}^T(\mu\mathbf{I} + \mathbf{QR})\mathbf{Q} = \mathbf{Q}^T\mathbf{AQ}.$$

Let G_μ denote the operation $\mathbf{A} \longrightarrow \mathbf{A}'$ defined by (13), (14): $\mathbf{A}' = G_\mu(\mathbf{A})$. We showed that the operator G_μ is commutative, in the sense $G_\mu G_\nu = G_\nu G_\mu$. Also, given Jacobi matrices $\mathbf{A}, \mathbf{A}'$ with $\sigma(\mathbf{A}) = \sigma(\mathbf{A}') = \sigma$, there is a unique set $(\mu_i)_1^{n-1}$ not to be confused with σ_n, such that

$$G_{\mu_1} G_{\mu_2} \ldots G_{\mu_{n-1}} \mathbf{A} = \mathbf{A}'. \tag{15}$$

In this procedure one goes from $\mathbf{A}$ to $\mathbf{A}'$, i.e., from P to P' on $\mathcal{S}$, by a sequence $n-1$ shifts along arcs $PQ_1, Q_1Q_2, \ldots, Q_{n-3}Q_{n-2}, Q_{n-2}P'$ on $\mathcal{S}$, as shown in Figure 2.

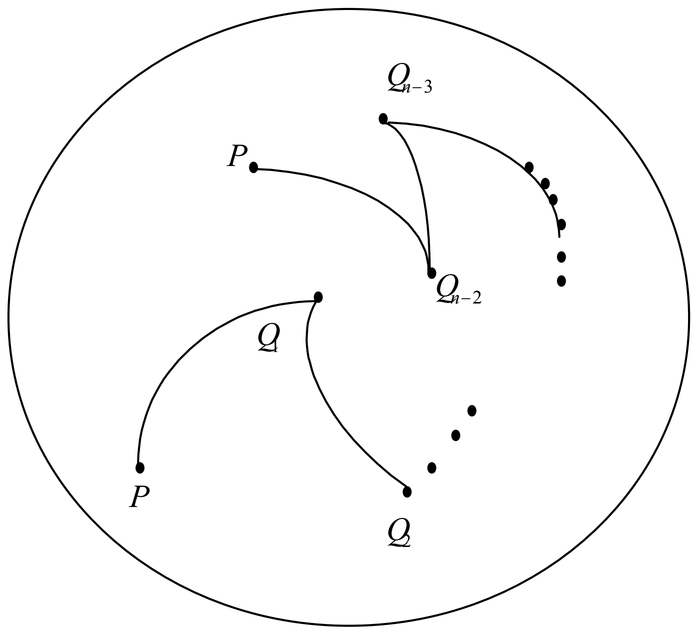


Figure 2 - The passage from P to P' along arcs PQ_1, Q_1Q_2, Q_2P' on the hypersphere S.

In these three procedures, $\mathbf{A}$ and $\mathbf{A}'$ correspond to points P, P' on $\mathcal{S}$ that are at a finite distance apart: the jump from P *to* P' is *discrete*. There is another way to generate an isospectral family, by considering an isospectral *flow*. All symmetric matrices $\mathbf{A}$ with $\sigma(\mathbf{A}) = \sigma = (\lambda_i)_1^n$ may be written

$$\mathbf{A} = \mathbf{Q}^T \wedge \mathbf{Q}, \tag{16}$$

where $\wedge = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Suppose that $\mathbf{Q}$ depends on a single parameter t. Denoting differentiation w.r.t. t by $\cdot$, we have

$$\begin{aligned} \dot{\mathbf{A}} &= \mathbf{Q}^T \wedge \dot{\mathbf{Q}} + \dot{\mathbf{Q}}^T \wedge \mathbf{Q}, \\ &= (\mathbf{Q}^T \wedge \mathbf{Q})(\mathbf{Q}^T \dot{\mathbf{Q}}) + (\dot{\mathbf{Q}}^T \mathbf{Q})(\mathbf{Q}^T \wedge \mathbf{Q}). \end{aligned} \tag{17}$$

But $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ implies $\mathbf{Q}^T\dot{\mathbf{Q}} + \dot{\mathbf{Q}}^T\mathbf{Q} = \mathbf{0}$, so that $\mathbf{Q}^T\dot{\mathbf{Q}} = \mathbf{S}, \dot{\mathbf{Q}}^T\mathbf{Q} = \mathbf{S}^T = -\mathbf{S}$: $\mathbf{S}$ is skew-symmetric. Thus $\mathbf{A}$ satisfies

$$\dot{\mathbf{A}} = \mathbf{AS} - \mathbf{SA} = [\mathbf{A}, \mathbf{S}], \tag{18}$$

where $[\mathbf{A}, \mathbf{S}]$ is the so-called Lie bracket. Any skew-symmetric $\mathbf{S}$ gives rise to a family of matrices $\mathbf{A}(t)$ which are isospectral. In the context of isospectral flow, we seek those matrices $\mathbf{S}$ which maintain some specified form of $\mathbf{A}(t)$.

In the simple case in which $\mathbf{A}$ is to remain a Jacobi matrix, we may take, as in Symes (1982),

$$\mathbf{A} = \begin{bmatrix} a_1 & -b_1 & & & \\ -b_1 & a_2 & -b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -b_{n-1} \\ & & & -b_{n-1} & a_n \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & b_1 & & & \\ -b_1 & 0 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & -b_{n-1} & 0 \end{bmatrix}. \tag{19}$$

This maintains the tridiagonal form of $\mathbf{A}$, and gives

$$\dot{a}_i = 2b_{i-1}^2 - 2b_i^2, \quad \dot{b}_i = (a_{i+1} - a_i)b_i, \tag{20}$$

where b_0, b_n are taken to be zero. We note that $a_{i+1} - a_i$ is some function of t, say $f(t)$, so that $b_i = f(t)b_i$, which has the solution

$$b_i(t) = C_i \exp(F(t)), \tag{21}$$

where $F(t) = \int_0^t f(t)dt$, and C_i is arbitrary. The function $f(t)$ is bounded for all t, by $\sum_{i=1}^n |\lambda_i|$, so that $b_i(t)$ retains the sign of C_i, i.e. of $b_i(0)$. If $\mathbf{A}(0)$ is a Jacobi matrix, with all $b_i(0)$ positive, then $\mathbf{A}(t)$ is a Jacobi matrix for all t.

Again, having generated $\mathbf{A}(t)$, we may reconstruct matrices $\mathbf{M}(t), \mathbf{K}(t)$ to give a system isospectral to $\mathbf{M}(0), \mathbf{K}(0)$. In the following sections we investigate whether and how these four ways of generating isospectral families may be generalized to types of matrices other than Jacobi matrices.

3. THE VIBRATING BEAM

There are many ways in which to construct a discrete model of a continuous beam vibrating in flexure. We choose to analyse a physical model of lumped masses connected by rigid rods and torsional springs, as described in Gladwell (1984). We will comment later on some possible finite element models. Since isospectral beams have recently been analysed in detail elsewhere, we merely sketch the treatment.

For definiteness, we consider the model of a cantilever, i.e., a clamped-free, beam, shown in Figure 3.

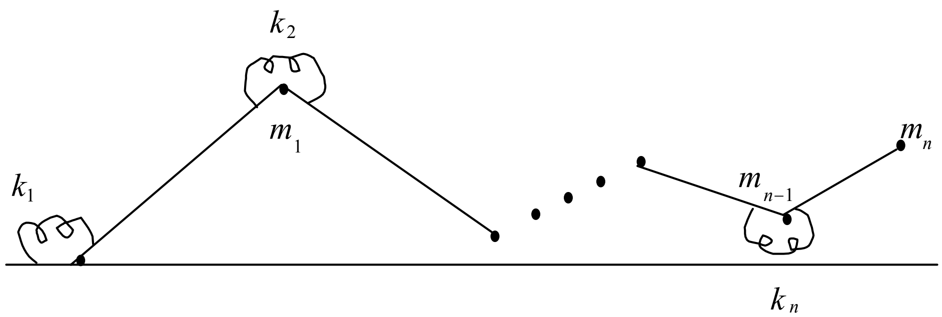


Figure 3 - A discrete model of a beam in flexure.

It may be shown that the stiffness and mass matrices are

$$\mathbf{K} = \mathbf{E}\mathbf{L}^{-1}\mathbf{E}\hat{\mathbf{K}}\mathbf{E}^{T}\mathbf{L}^{-1}\mathbf{E}^{T}, \quad \mathbf{M} = diag(m_1, m_2, \ldots, m_n), \tag{22}$$

where

$$\mathbf{L} = diag(l_1, l_2, \ldots, l_n), \ \hat{\mathbf{K}} = diag(k_1, k_2, \ldots, k_n), \tag{23}$$

and $\mathbf{E}$ is given by (7). The alternating signs of $\mathbf{E}$ lead to alternating signs in the symmetric pentadiagonal matrix $\mathbf{K}$; k_{ij} has the sign of $(-1)^{i+j}$. Now the MRK is

$$\mathbf{A} = \mathbf{D}^{-1}\mathbf{K}\mathbf{D}^{-1} \tag{24}$$

where $\mathbf{M} = \mathbf{D}^2$, $\mathbf{D} = diag(d_1, d_2, \ldots, d_n)$. The matrix $\mathbf{A}$ is pentadiagonal with alternating signs. However, this description of $\mathbf{A}$ is insufficient; given a symmetric pentadiagonal matrix $\mathbf{A}$ with alternating signs, it is in general not possible to write it in the form dictated by (22), (23), for real positive values of the parameters $(k_i, l_i, m_i)_1^n$. Nor is it enough for $\mathbf{A}$ to be PD. The necessary and sufficient conditions for $\mathbf{A}$ to be PD are that the *leading principal minors* of $\mathbf{A}$ are positive:

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots \quad \det(A) > 0. \tag{25}$$

The requisite conditions for factorization as in (22), (24), involve *all* the minors of $\mathbf{A}$; the necessary and sufficient condition is that $\mathbf{A}$ is what is called a *sign-oscillatory* (SO) matrix, as we now describe.

Suppose $\mathbf{B}$ is a square matrix of order n, i.e. $\mathbf{B} \in M_n$; $\mathbf{B}$ need not be symmetric. A minor of $\mathbf{B}$ of order p is the determinant of the matrix formed from p rows and p columns of $\mathbf{B}$. Define the diagonal matrix $Z = diag(+1, -1, +1, \ldots, \pm 1)$. If $\mathbf{A}$ has alternate signs, then $\mathbf{B} = \mathbf{ZAZ}$ has all the same signs; $b_{ij} = (-1)^{i+j} a_{ij}$.

Now we are ready for some definitions:

1. $\mathbf{B}$ is *totally positive* (TP) if all its minors are non-negative.

2. NTP if it is non-singular and TP

3. STP (strictly TP) if all its minors are strictly positive

4. O (oscillatory) if $\mathbf{B}$ is TP, and $\mathbf{B}^m$ is STP for some positive integer m. It is known that $\mathbf{B}$ is O iff $\mathbf{B}$ is NTP and $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for $i = 1, 2, \ldots, n-1$

5. $\mathbf{A}$ is said to be SO (sign-oscillatory) if $\mathbf{B} = \mathbf{ZAZ}$ is O.

Clearly, the MRK for the beam is SO; the pre- and post-multiplication of $\mathbf{A}$ by $\mathbf{Z}$ suppresses the signs, all the minors of $\mathbf{B} = \mathbf{ZAZ}$ will be non-negative, $\mathbf{B}$ is non-singular, and as may easily be verified, the terms next to the principal diagonal are strictly positive.

We have shown in Gladwell (2001a) that when $\mathbf{A}$ is symmetric, pentadiagonal and SO, with all the terms in the five diagonals of $\mathbf{B} = \mathbf{ZAZ}$ strictly positive, then $\mathbf{A}$ may be factorized as in (22), (24), with positive $(k_i, l_i, m_i)_1^n$. Now we ask how, given one such $\mathbf{A}$ specified by $(k_i, l_i, m_i)_1^n$, we can find another $\mathbf{A}'$ isospectral to $\mathbf{A}$, and corresponding to other (positive) $(k_i', l_i', m_i')_1^n$.

On the basis of the analysis in Section 2, we can list four candidate procedures:

i) It is known (Gladwell (1984)) that the beam may be reconstructed, apart from two scaling factors, from three spectra $(\lambda_i)_1^n, (\mu_i)_1^{n-1}$ and $(\sigma_i)_1^{n-1}$ corresponding to the right hand end being free, pinned or sliding, respectively. Thus, one possibility is to retain $(\lambda_i)_1^n$, and change $(\mu_i)_1^{n-1}$ and $(\sigma_i)_1^{n-1}$.

ii) Construct $\mathbf{A}$ from its spectrum $(\lambda_i)_1^n$ and the values $(u_n^{(i)})_1^n$, $(u_{n-1}^{(i)})_1^n$ of the last two elements of its normalized eigenvectors.

iii) Carry out the G_μ operation defined by shifted QR factorization and reversal.

iv) Set up an isospectral flow equation. We now comment on these four candidates:

i) For the rod, just two spectra, $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ were needed, and the necessary and sufficient condition for realizability was the interlacing (11). For the beam there are three spectra, $(\lambda_i)_1^n, (\mu_i)_1^{n-1}$ and $(\sigma_i)_1^{n-1}$. One can show that they interlace according to

$$\lambda_1 < \sigma_1 < \mu_1 < \lambda_2 < \sigma_2 < \mu_2 < \cdots < \sigma_{n-1} < \mu_{n-1} < \lambda_n. \tag{26}$$

Thus, the clamped-sliding natural frequencies are always less than the clamped-pinned natural frequencies. *BUT* now this interlacing is not sufficient for realizability, i.e., for the k_i, l_i, m_i to be positive.

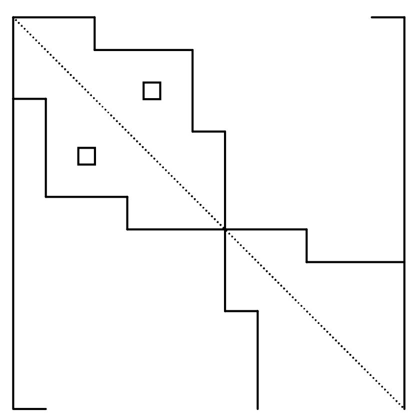
ii) The necessary and sufficient conditions may be phrased in terms of the $(u_n^{(i)})_1^n$ and $(u_{n-1}^{(i)})_1^n$, or equivalently in terms of $(u_n^{(i)})_1^n$ and $(\theta_n^{(i)})_1^n$, where $\theta_n^{(i)}$ denotes the angle that the last rod, joining m_{n-1} and m_n, makes with the axis. The conditions are that a certain matrix constructed from $(\lambda_i, u_n^{(i)}, \theta_n^{(i)})_1^n$ is strictly totally positive, as described in Gladwell (1984). The consequence of this is that it is extremely difficult, in fact well nigh impossible, to choose the data in procedures i) or ii) to satisfy all these positivity conditions.

iii) This procedure is feasible. Gladwell (1998) showed that if $\mathbf{A}$ is symmetric and either NTP, STP, O or SO, if μ is not an eigenvalue of $\mathbf{A}$, then $\mathbf{A}' = G_\mu \mathbf{A}$, defined by (13), (14) has the same property NTP, STP, O or SO respectively. Once we have chosen μ, and constructed $\mathbf{A}'$ it is comparatively simple to factorize $\mathbf{A}' = (\mathbf{D}')^{-1}\mathbf{K}'(\mathbf{D}')^{-1}$, where $\mathbf{K}' = \mathbf{E}(\mathbf{L}')^{-1}\mathbf{E}\hat{\mathbf{K}}'\mathbf{E}^T(\mathbf{L}')^{-1}\mathbf{E}^T$, and $\hat{\mathbf{K}}', \mathbf{L}', \mathbf{D}'$ are diagonal.

iv) This procedure is feasible too. Recently, Gladwell (2001) has shown that if $\mathbf{A}(0)$ is symmetric and either TP, NTP, STP, O or SO, and $\mathbf{A}(t)$, depending on a single parameter t, flows according to (18), for a certain $\mathbf{S}$, then $\mathbf{A}(t)$ retains the same property TP, NTP, STP, O or SO.

This result holds not just for pentadiagonal matrices, but for general symmetric matrices. However, the total positivity property forces $\mathbf{A}$ to have a so-called *staircase* form. In a symmetric staircase matrix, all the terms are zero outside a staircase structure around the principal diagonal.

(a)



(b)

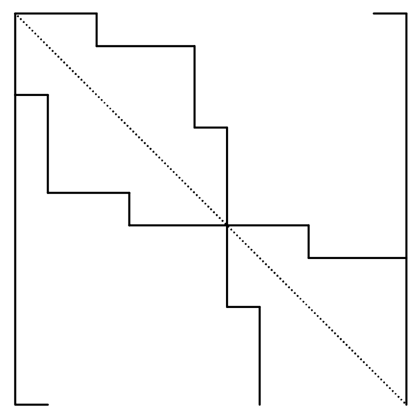


Figure 4 - A staircase matrix (a) with and (b) without holes.

A general symmetric staircase matrix may have holes, i.e., there may be zero terms inside the staircase as shown in Figure 4(a). A symmetric NTP matrix is always a staircase with no holes, as in Figure 4(b).

The matrix $\mathbf{S}$ for which this isospectral flow retains these total positivity properties is the generalization of that in (19):

$$S = \begin{bmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ a_{12} & 0 & -a_{23} & \cdots & -a_{2n} \\ a_{13} & a_{23} & 0 & \cdots & -a_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & a_{3n} & \cdots & 0 \end{bmatrix}. \tag{27}$$

If $\mathbf{A}^+$ denotes the upper triangle of $\mathbf{A}$, then $\mathbf{S} = \mathbf{A}^{+T} - \mathbf{A}^+$. Clearly, $\mathbf{S}$ is skew symmetric. Not only does the flow (18) retain the positivity properties, but it also retains band properties: if $\mathbf{A}(0)$ is pentadiagonal, then $\mathbf{A}(t)$ is pentadiagonal. Moreover, if $\mathbf{A}(0)$ has all the terms in its outer diagonals non-zero, then so will $\mathbf{A}(t)$. Thus, once we have constructed $\mathbf{A}(t)$, we may factorize it to give $\mathbf{K}(t), \mathbf{L}(t), \mathbf{D}(t)$, as in iii).

4. FINITE ELEMENT MODELS

These models are much more challenging than the comparatively simple physical models that we have considered until now, in this paper. The simplest F.E. model is that for a thin straight rod in longitudinal vibration. Use of two linear assumed modes for each element leads to stiffness and mass matrices which are both tridiagonal. $\mathbf{K}$ is symmetric, PSD (or PD if the rod is anchored), and has negative co-diagonal; $\mathbf{K}$ is SO if it is PD. The mass matrix $\mathbf{M}$ is PD and has positive co-diagonal; it is O. What does the MRK look like?

We factorize $\mathbf{M} = \mathbf{L}\mathbf{L}^T$; $\mathbf{L}$ is lower bidiagonal with positive terms; it is a so-called lower triangular oscillatory matrix. Its inverse, $\mathbf{L}^{-1}$ is again lower triangular full, and has alternating signs; it is lower triangular sign-oscillatory. It may be verified that the MRK, namely $\mathbf{A} = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$ is a full matrix, and is SO.

Gladwell (1995) showed that $\mathbf{A}' = G_\mu \mathbf{A}$, which is known to be SO, can in fact be factorized as $\mathbf{A}' = (\mathbf{L}')^{-1}\mathbf{K}'(\mathbf{L}')^{-T}$ where $\mathbf{K}'$ is tridiagonal and SO, and $\mathbf{M} = \mathbf{L}'\mathbf{L}'^T$ is tridiagonal and O. The details of the proof, which is intricate, are given in Gladwell (1995).

One of the matters which arises in Gladwell (1995), and which is addressed in Gladwell (1999), is this: how can we construct *one* pair of matrices $\mathbf{M}, \mathbf{K}$ of the required form so that $\sigma(\mathbf{M}, \mathbf{K}) = (\lambda_i)_1^n$. We proceed as follows. One spectrum is insufficient for the reconstruction of a Jacobi matrix, so it is clearly insufficient for the reconstruction of the two tridiagonal matrices $\mathbf{M}$ and $\mathbf{K}$. We therefore reduce the unknowns in the problem by seeking an $\mathbf{M}$ which can be expressed in terms of $\mathbf{K}$ by an equation of the form

$$\mathbf{M} = \mathbf{D}^2 - c\mathbf{K}, \tag{28}$$

where $\mathbf{D}$ is diagonal and c is a positive scalar; if $\mathbf{K}$ has negative co-diagonal, then $\mathbf{M}$ will have positive co-diagonal. Then

$$\mathbf{K} - \lambda\mathbf{M} = \mathbf{K} - \lambda(\mathbf{D}^2 - c\mathbf{K}) = (1 + c\lambda)(\mathbf{K} - \nu D^2) \tag{29}$$

where $\nu = \lambda/(1 + c\lambda)$. Now we reduce $(\mathbf{K} - \nu\mathbf{D}^2)\mathbf{u} = 0$ to $(A - \nu I)\mathbf{x} = 0$ as before. If $\sigma(M, K) = (\lambda_i)_1^n$ then $\sigma(\mathbf{A}) = (\nu_i)_1^n)$, where $\nu_i = \lambda_i/(1 + c\lambda_i)$. We note that if the eigenvectors of $\mathbf{A}$ are the columns of

$$\mathbf{X} = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}] \tag{30}$$

then $\mathbf{AX} = \mathbf{XN}$, where $\mathbf{N} = diag(\nu_1, \nu_2, \ldots, \nu_n)$. Thus $\mathbf{A} = \mathbf{XNX}^T$, and

$$\begin{aligned} \mathbf{M} &= \mathbf{D}^2 - c\mathbf{K} = \mathbf{D}^2 - c\mathbf{DAD} = \mathbf{D}(\mathbf{I} - c\mathbf{A})\mathbf{D}, \\ &= \mathbf{D}(\mathbf{I} - c\mathbf{XNX}^T)\mathbf{D} = \mathbf{DX}(\mathbf{I} - c\mathbf{N})\mathbf{X}^T\mathbf{D}, \end{aligned} \tag{31}$$

and $\mathbf{I} - c\mathbf{N}$ is diagonal with elements $1 - c\nu_i = 1/(1 + c\lambda_i)$, so that $\mathbf{M}$ is PD.

This is fact provides *another* way to find an isospectral system $\mathbf{M}', \mathbf{K}'$:

i) Given $(\lambda_i)_1^n$, choose $(\mu_i)_1^{n-1}$ interlacing $(\lambda_i)_1^n$, as in (11). Choose $c > 0$, and construct $\mathbf{A}$ such that $\sigma(\mathbf{A}) = (\nu_i)_1^n$, $\sigma_n(\mathbf{A}) = (\kappa_i)_1^{n-1}$, where $\kappa_i = \mu_i/(1 + c\mu_i)$. (If $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ interlace, so will $(\nu_i)_1^n$ and $(\kappa_i)_1^{n-1}$.)

ii) From $\mathbf{A}$ construct $\mathbf{A}'$, using either the shifted QR, RQ factorization, or isospectral flow.

iii) Find $\mathbf{D}'$ so that $\mathbf{K}' = \mathbf{D}'\mathbf{A}'\mathbf{D}'$ satisfies the characteristic equation (9) for a stiffness matrix; and note that if the stiffness matrix is constructed from linear assumed modes, then (9) will still hold.

iv) Form $\mathbf{M}' = \mathbf{D}'^2 - c\mathbf{K}'$; $\mathbf{M}'$ will be PD, tridiagonal, and have positive co-diagonal.

5. SOME OPEN PROBLEMS

All the systems that have been discussed in this paper have been, in a sense, one-dimensional: they have concerned rods and beams—linear objects. In each case, the mass and stiffness matrices have had a staircase form, and it is staircase matrices which retain their form under QR⟶RQ transformation, or under the isospectral flow (18). Thus, the quest for isospectral pairs of matrices relating to 2- or 3-dimensional objects - membranes, plates, etc. - remains open.

There are of course still open problems relating to one-dimensional objects. One of them is this. The pentadiagonal matrix appearing in the beam problem can be reconstructed from three suitably chosen spectra. Thus, there is an (infinite) family of isospectral pentadiagonal matrices with two spectra in common. If these two spectra are σ and σ_1 then two such matrices A, A' must be related by an orthogonal transformation of the form

$$\mathbf{A}' = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^T \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix}, \tag{32}$$

where $\mathbf{Q}$ is an orthogonal matrix in $\mathbf{M}_{n-1}$. Writing

$$\mathbf{A} = \begin{bmatrix} a_1 & \mathbf{b}^T \\ \mathbf{b} & \mathbf{B} \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_1 & \mathbf{b}'^T \\ \mathbf{b}' & \mathbf{B}' \end{bmatrix} \tag{33}$$

where $\mathbf{b}^T = [b_1, c_1, 0, \ldots, 0]$, $\mathbf{b}'^T = [b'_1, c'_1, 0, \ldots, 0]$, we find

$$\mathbf{b}' = \mathbf{Q}^T\mathbf{b}, \quad \mathbf{B}' = \mathbf{Q}^T\mathbf{B}\mathbf{Q}. \tag{34}$$

The first equation places constraints on the columns of $\mathbf{Q}$, and it is not clear how these constraints should be satisfied. The study of isospectral systems still provides many challenges.

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