Total positivity and Toda flow

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Abstract

A real matrix \(A \in M_n\) is TP (totally positive) if all its minors are non-negative; NTP, if it is non-singular and TP; STP, if it is strictly TP; O (oscillatory) if it is TP and a power \(A^m\) is STP. We consider the Toda flow of a symmetric matrix \(A(t)\), and show that if \(A(0)\) is one of TP, NTP, STP or O, then \(A(t)\) is TP, NTP, STP or O, respectively. © 2002 Elsevier Science Inc. All rights reserved.

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1. Total positivity

Total positivity properties play an important role in the characterisation of matrices which appear in the vibration of mechanical systems, as described in [6] or [8]. We recall some definitions. A matrix \(A \in M_n\) is said to be
(i) TP (totally positive) if all its minors are non-negative.
(ii) NTP if it is non-singular and TP.
(iii) STP (strictly TP) if all its minors are strictly positive.
(iv) O (oscillatory) if it is TP and a power \(A^m\) is STP, for \(1 \leq m \leq n - 1\).
(v) SO (sign-oscillatory) if the matrix \(\tilde{A} = TAT\) is O, where \(T = \text{diag}(+1, -1, +1, \ldots, \pm 1)\).

It is known that \(A\) is O iff it is NTP and the diagonals next to the principal diagonal are strictly positive: \(a_{i,i+1} > 0\), \(a_{i+1,i} > 0\), \(i = 1, 2, \ldots, n - 1\). Thus, a symmet-
ric, positive definite, tridiagonal matrix is O (SO) if its codiagonal is strictly positive (negative). It is known also that $A$ is O (SO) iff $A^{-1}$ is SO (O).

In a recent paper, Gladwell [9] proved that for symmetric (real) matrices, $A \in S_n$, the properties NTP, STP and O are preserved under shifted QR transformation. Theorem 2.1 of that paper may be rephrased as follows: suppose $A \in S_n$ and $A$ have one of the properties TP, NTP, STP, O, SO; $\mu$ is not an eigenvalue of $A$; $A - \mu I = QR$ where $Q$ is orthogonal and $R$ is upper triangular with positive diagonal; $A' - \mu I = RQ'$; then $A'$ is TP, NTP, STP, O, SO, respectively.

The proof of this result stems from a second fundamental relation between $A$ and $A'$:

$$RA = A'R,$$

and the corresponding relation

$$R_pA_p = A'_pR_p$$

between the $p$th compound matrices, derived from the Cauchy–Binet theorem.

It is well known [1] that $A \in M_n$ is STP iff all the minors taken from successive rows and columns of $A$ are strictly positive. This result is due to Fekete [5]. It is shown in [7,9] that Eqs. (1) and (2) yield an important test for $A$ to be STP if it is known that $A$ is NTP: $A$ is STP iff the lower left and upper right corner minors of $A$ are strictly positive. Thus in the notation of Ando [1],

$$A[1, 2, \ldots, p \mid n - p + 1, \ldots n] > 0, \quad p = 1, 2, \ldots, n,$$

(3)

$$A[n - p + 1, \ldots, n \mid 1, 2, \ldots, p] > 0, \quad p = 1, 2, \ldots, n.$$  

(4)

Of course if $A \in S_n$, the minors in (3) and (4) are identical.

Ando [1] shows that the STP matrices are dense in the set of TP matrices. Specifically, if $A$ is TP, then $C(k) = P(k)[A + \exp(-k)I]P(k)$ is STP, where

$$P(k) = (p_{ij}), \quad p_{ij} = \exp[-k(i - j)^2]$$

(5)

and $k = 1, 2, 3, \ldots$ Clearly

$$C(k) = A + O(\exp(-k)),$$

(6)

so that $A$ may be approximated arbitrarily closely in, say, the $L_1$ or the Frobenius norm, by the STP matrix $C(k)$.

2. Toda flow

We consider the Toda flow

$$\dot{A} = AS - SA = [A, S]$$

(7)

for $A \equiv A(t) \in S_n$, where $S = A^+T - A^+$, and $A^+$ is the upper triangle of $A$. We will show that Toda flow preserves certain total positivity properties. Guided by the
considerations in Section 1, we pay particular attention to the corner minors of $A$ and powers of $A$.

First we note that if $B = A^m$, $m = 1, 2, 3 \ldots$, then $B$ satisfies the same equation as $A$:

$$\dot{B} = BS - SB,$$  \hspace{1cm} (8)

where $S = A^+T - A^+$. Since we must find how the minors of $B$ behave, we need a lemma on determinants:

**Lemma 1.** Suppose $b_1, b_2, \ldots, b_p \in V_p$ are the columns of $B_p \in M_p$; $C \in M_p$, $d_i = Cb_i$, then

$$\sum_{j=1}^{p} \det(b_1, \ldots, b_{j-1}, d_j, b_{j+1}, \ldots, b_p) = \text{tr}(C) \det B_p.$$  

**Proof.** This follows immediately from the fact that if $A \in M_p$ and $A_{ij}$ are the cofactors of $a_{ij}$, then

$$\sum_{i=1}^{p} a_{ij} A_{ik} = \delta_{jk} \det(A).$$

Now we prove

**Theorem 1.** Suppose $A(t) \in S_n$ satisfies (7), $B = A^m$, $c_p = B[1, 2, \ldots, p|n - p + 1, \ldots, n]$, then $c_p(t)$ satisfies

$$\dot{c}_p = \left(\sum_{j=n-p+1}^{n} a_{jj} - \sum_{j=1}^{p} a_{jj}\right) c_p, \hspace{0.5cm} p = 1, 2, \ldots, n.$$  \hspace{1cm} (9)

**Proof.** Denote the $p$th order corner matrix of $A$ by $B_p$, and suppose its columns are $b_1, b_2, \ldots, b_p \in V_p$. Thus

$$b_j = [b_{n-p+1, j}, b_{n-p+2, j}, \ldots, b_{n, j}]^T.$$  

Eq. (8) gives

$$\dot{b}_{ij} = (a_{ii} - a_{jj})b_{ij} - 2\sum_{k=1}^{j-1} a_{jk}b_{ik} + 2\sum_{k=i+1}^{n} a_{ik}b_{kj}$$

so that

$$\dot{b}_j = -a_{jj}b_j - 2\sum_{k=1}^{j-1} a_{jk}b_k + Cb_j.$$  \hspace{1cm} (10)
where $C \in M_p$ is given by
\[ c_{ik} = \begin{cases} a_{ii}, & i = k, \\ 2a_{ik}, & k = i + 1, \ldots, n \end{cases} \]
for $i, k = n - p + 1, \ldots, n$.

Now $c_p = \det(b_1, b_2, \ldots, b_p)$, so that
\[ \dot{c}_p = \sum_{j=1}^{p} \det(b_1, b_2, \ldots, b_{j-1}, b_j, b_{j+1}, \ldots, b_p). \] (11)

Consider the sums obtained by inserting each of the three terms in $\dot{b}_j$ from (10) into (11). The first gives
\[ -\sum_{j=1}^{p} a_{jj}c_p. \]

The second gives zero because it is merely a combination of the first $j - 1$ columns. The lemma gives the third as
\[ \sum_{j=n-p+1}^{n} a_{jj}c_p. \] \qed

We now prove

**Theorem 2.** Suppose $A(0) \in S_n$ has any one of the properties TP, NTP, STP, O, SO, then $A(t)$ has the corresponding property for all $t$.

**Proof.** Theorem 1 shows that all the corner minors of $A$, and of $B = A^m$, satisfy a differential equation of the form
\[ \dot{y} = g(t)y, \] (12)
where $g(t)$, given by (9), is continuous and bounded by $|g(t)| \leq \text{tr}(A(t)) = \text{tr}(A(0))$.
Under these conditions, the solution of (12) is
\[ y(t) = y(0) \exp(G(t)), \]
where $G(t) = \int_0^t g(t) \, dt$. Thus $y(t)$ retains its sign over $(-\infty, \infty)$, i.e., $y(t)$ has the same sign as $y(0)$. Now consider the various cases:

(a) $A(0)$ is STP. By continuity there is an interval $(a, b)$ around 0 in which $A(t)$ is STP. Suppose if possible that, as $t$ increases from zero, one or more minors of $A$ first become zero at $t = b$. Then $A(b)$ is merely NTP. But $A(0)$ is STP, so that $c_p(0) > 0$, $p = 1, 2, \ldots, n$ and thus $c_p(b) > 0$. But then $A(b)$ is NTP and has strictly positive corner minors: $A(b)$ is STP. This contradiction implies $A(t)$ is STP for all $t > 0$; an exactly similar argument shows that $A(t)$ is STP for all $t < 0$. $A(t)$ is STP for all $t$. 


(b) $A(0)$ is TP. For every $k = 1, 2, \ldots$, $C(k, 0) = P(k)\{A(0) + \exp(-k)I\}P(k)$ is STP. Under the Toda flow (7), $C(k, t)$ remains STP. Now we apply standard results on the variation of solutions of o.d.e.'s with respect to initial conditions and parameters, e.g. [4, Theorem 4.1]. The Toda flow equation may be written

$$\dot{A} = f(A).$$

The matrix $f(A(t))$ is bounded. In the Frobenius norm

$$||f(A(t))||_2 \leq 2||A(t)||_2^2 = 2||A(0)||_2^2$$

so that

$$\lim_{k \to \infty} C(k, 0) = A(0)$$

implies

$$\lim_{k \to \infty} C(k, t) = A(t).$$

But $C(k, t)$ is STP so that $\lim_{k \to \infty} C(k, t)$ is TP. Thus $A(t)$ is TP.

(c) $A(0)$ is NTP. $A(0)$ is TP, so that $A(t)$ is TP, $\det(A(t)) = \det(A(0)) \neq 0$, so that $A(t)$ is TP.

(d) $A(0)$ is O. First $A(0)$ is NTP, so that $A(t)$ is NTP, and hence $B(t) = A^m(t)$ is NTP, for all $m = 1, 2, \ldots$. Secondly, for some $m$, with $1 \leq m \leq n - 1$, $B(0) = A^m(0)$ is STP so that, by (a), $B(t)$ is STP: $A(t)$ is O.

(e) $A(0)$ is SO; $\tilde{A}(0)$ is O; $\tilde{A}(t)$ is O; $A(t)$ is SO. \(\square\)

3. Concluding remarks

In this paper we have been concerned with signs, with patterns of signs which are preserved under Toda flow. There is a large body of research connected with matrix shapes, i.e., patterns of zero and non-zero terms, that are preserved under QR transformation [2] or under a wide class of Toda-like flows [3]. The basic pattern that is preserved, in both cases, is the staircase.

A sequence $p = \{p_1, p_2, \ldots, p_n\}$ is said to be a staircase sequence if $1 \leq p_1 \leq p_2 \leq \cdots \leq p_n \leq n$ and $p_i \geq i$, $i = 1, 2, \ldots, n$. A symmetric matrix is $p$-staircase if $a_{ij} = 0$ for $j > p_i$, $i = 1, 2, \ldots, n$. For a strict band matrix with half-band width $r$, $p_i = i + r$. In that case $p_1 < p_2 < \cdots < p_{n-r}$.

The Toda flow (7) has the property that if $A$ is a symmetric $p$-staircase matrix, and $a_{ij}$ lies outside the staircase, then $\dot{a}_{ij} = 0$. Thus, terms initially outside the staircase remain zero. It may easily be verified that if $a_{ij}$ is on the tip of a stair of the staircase i.e., $j = p_i$ and either $i = 1$ or $p_i > p_{i-1}$, then $\dot{a}_{ij} = (a_{jj} - a_{ii})a_{ij}$, so that if $a_{ij}(0) > , =, < 0$ then $a_{ij}(t) > , =, < 0$, respectively. Other terms on the boundary of the staircase may become zero or change their signs. If $A(0)$ is a strict band matrix with half-band width $r$, so that all the terms in the outermost diagonal are non-zero, then since $p_1 < p_2 < \cdots < p_{n-r}$, all the terms on that diagonal retain their signs.
The definition of a staircase matrix does not state that $a_{ij} \neq 0$ for $i \leq j \leq p_i$; there may be zero terms, i.e., holes, in the staircase. For a general symmetric matrix under Toda flow, even if $A(0)$ is a $p$-staircase with no holes, then holes may appear in the $p$-staircase for $A(t)$.

What differences occur when $A(0)$ has some TP property? Markham [10] effectively showed that if $A \in S_n$ is O, then $A$ is a $p$-staircase with no holes: all the terms inside and on the staircase are positive. It may be verified that this result still holds even if $A(0)$ is merely NTP. This means that if $A(0)$ is NTP, so that $A(0)$ is a $p$-staircase with no holes, then $A(t)$ will remain the same $p$-staircase pattern, and all the terms inside and on the staircase will remain positive.

We have deliberately phrased the problem and the results in the simplest form; undoubtedly they may be generalised in many ways, but we leave this to others.

References