LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER Linear Algebra and its Applications 350 (2002) 279–284

www.elsevier.com/locate/laa

Total positivity and Toda flow

G.M.L. Gladwell

Department of Civil Engineering, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

Received 30 August 2001; accepted 9 February 2002

Submitted by H. Schneider

Abstract

A real matrix $A \in M_n$ is TP (totally positive) if all its minors are nonnegative; NTP, if it is non-singular and TP; STP, if it is strictly TP; O (oscillatory) if it is TP and a power A^m is STP. We consider the Toda flow of a symmetric matrix $A(t)$, and show that if $A(0)$ is one of TP, NTP, STP or O, then $A(t)$ is TP, NTP, STP or O, respectively. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: 15A48, 58J30, 58J53

Keywords: Totally positive matrix; Oscillatory matrix; Toda flow

1. Total positivity

Total positivity properties play an important role in the characterisation of matrices which appear in the vibration of mechanical systems, as described in [6] or [8]. We recall some definitions. A matrix $A \in M_n$ is said to be
- (i) TP (totally positive) if all its minors are non-negative.
- (ii) NTP if it is non-singular and TP.
- (iii) STP (strictly TP) if all its minors are strictly positive.
- (iv) O (oscillatory) if it is TP and a power A^m is STP, for $1 \leqslant m \leqslant n-1$.
- (v) SO (sign-oscillatory) if the matrix $\tilde{A} = TAT$ is O, where $T = \mathrm{diag}(+1, -1, +1, \ldots, \pm 1)$.

It is known that A is O iff it is NTP and the diagonals next to the principal diagonal are strictly positive: $a_{i,i+1} > 0,\ a_{i+1,i} > 0,\ i = 1, 2, \ldots, n-1$. Thus, a symmet-

E-mail address: ggladwell@uwaterloo.ca (G.M.L. Gladwell).

URL: http://www.civil.uwaterloo.ca//ggladwell.

PII: S0024-3795(02)00331-2

ric, positive definite, tridiagonal matrix is O (SO) if its codiagonal is strictly positive (negative). It is known also that A is O (SO) iff A^{-1} is SO (O).

In a recent paper, Gladwell [9] proved that for symmetric (real) matrices, $A \in S_n$, the properties NTP, STP and O are preserved under shifted *QR* transformation. Theorem 2.1 of that paper may be rephrased as follows: suppose $A \in S_n$ and A have one of the properties TP, NTP, STP, O, SO; μ is not an eigenvalue of A; $A - \mu I = QR$ where Q is orthogonal and R is upper triangular with *positive* diagonal; $A' - \mu I = RQ$; then A' is TP, NTP, STP, O, SO, respectively.

The proof of this result stems from a second fundamental relation between A and A':

$$RA = A'R, \tag{1}$$

and the corresponding relation

$$\mathscr{R}_p \mathscr{A}_p = \mathscr{A}'_p \mathscr{R}_p \tag{2}$$

between the pth compound matrices, derived from the Cauchy–Binet theorem.

It is well known [1] that $A \in M_n$ is STP iff all the minors taken from *successive* rows and columns of A are strictly positive. This result is due to Fekete [5]. It is shown in [7,9] that Eqs. (1) and (2) yield an important test for A to be STP *if it is known that A is NTP*: A is STP iff the lower left and upper right corner minors of A are strictly positive. Thus in the notation of Ando [1],

$$A[1, 2, \ldots, p \,|\, n - p + 1, \ldots n] > 0, \quad p = 1, 2, \ldots, n, \tag{3}$$

$$A[n - p + 1, \ldots, n \,|\, 1, 2, \ldots, p] > 0, \quad p = 1, 2, \ldots, n. \tag{4}$$

Of course if $A \in S_n$, the minors in (3) and (4) are identical.

Ando [1] shows that the STP matrices are dense in the set of TP matrices. Specifically, if A is TP, then $C(k) = P(k)\{A + \exp(-k)I\}P(k)$ is STP, where

$$P(k) = (p_{ij}), \quad p_{ij} = \exp[-k(i - j)^2] \tag{5}$$

and $k = 1, 2, 3, \ldots$ Clearly

$$C(k) = A + \mathrm{O}(\exp(-k)), \tag{6}$$

so that A may be approximated arbitrarily closely in, say, the L_1 or the Frobenius norm, by the STP matrix $C(k)$.

2. Toda flow

We consider the Toda flow

$$\dot{A} = AS - SA = [A, S] \tag{7}$$

for $A \equiv A(t) \in S_n$, where $S = A^{+\mathrm{T}} - A^{+}$, and A^{+} is the upper triangle of A. We will show that Toda flow preserves certain total positivity properties. Guided by the considerations in Section 1, we pay particular attention to the corner minors of A and powers of A.

First we note that if $B = A^m$, $m = 1, 2, 3 \ldots$, then B satisfies the same equation as A:

$$\dot{B} = BS - SB, \tag{8}$$

where $S = A^{+\mathrm{T}} - A^{+}$. Since we must find how the minors of B behave, we need a lemma on determinants:

Lemma 1. *Suppose $b_1, b_2, \ldots, b_p \in V_p$ are the columns of $B_p \in M_p$; $C \in M_p$, $d_i = Cb_i$, then*

$$\sum_{j=1}^{p} \det(b_1, \ldots, b_{j-1}, d_j, b_{j+1}, \ldots, b_p) = \operatorname{tr}(C) \det B_p.$$

Proof. This follows immediately from the fact that if $A \in M_p$ and A_{ij} are the cofactors of a_{ij}, then

$$\sum_{i=1}^{p} a_{ij} A_{ik} = \delta_{jk} \det(A). \qquad \square$$

Now we prove

Theorem 1. *Suppose $A(t) \in S_n$ satisfies (7), $B = A^m$, $c_p = B[1, 2, \ldots, p|n - p + 1, \ldots, n]$, then $c_p(t)$ satisfies*

$$\dot{c}_p = \left(\sum_{j=n-p+1}^{n} a_{jj} - \sum_{j=1}^{p} a_{jj} \right) c_p, \quad p = 1, 2, \ldots, n. \tag{9}$$

Proof. Denote the pth order corner matrix of A by B_p, and suppose its columns are $b_1, b_2, \ldots, b_p \in V_p$. Thus

$$b_j = [b_{n-p+1,j}, b_{n-p+2,j}, \ldots, b_{n,j}]^{\mathrm{T}}.$$

Eq. (8) gives

$$\dot{b}_{ij} = (a_{ii} - a_{jj}) b_{ij} - 2 \sum_{k=1}^{j-1} a_{jk} b_{ik} + 2 \sum_{k=i+1}^{n} a_{ik} b_{kj}$$

so that

$$\dot{b}_j = -a_{jj} b_j - 2 \sum_{k=1}^{j-1} a_{jk} b_k + C b_j, \tag{10}$$

where $C \in M_p$ is given by

$$c_{ik} = \begin{cases} a_{ii}, & i = k, \\ 2a_{ik}, & k = i + 1, \dots, n \end{cases}$$

for $i, k = n - p + 1, \dots, n$.

Now $c_p = \det(b_1, b_2, \dots, b_p)$, so that

$$\dot{c}_p = \sum_{j=1}^{p} \det(b_1, b_2, \dots, b_{j-1}, \dot{b}_j, b_{j+1}, \dots, b_p). \tag{11}$$

Consider the sums obtained by inserting each of the three terms in $\dot{b}_j$ from (10) into (11). The first gives

$$-\sum_{j=1}^{p} a_{jj} c_p.$$

The second gives zero because it is merely a combination of the first $j - 1$ columns. The lemma gives the third as

$$\sum_{j=n-p+1}^{n} a_{jj} c_p. \qquad \square$$

We now prove

Theorem 2. *Suppose $A(0) \in S_n$ has any one of the properties TP, NTP, STP, O, SO, then $A(t)$ has the corresponding property for all t.*

Proof. Theorem 1 shows that all the corner minors of A, and of $B = A^m$, satisfy a differential equation of the form

$$\dot{y} = g(t) y, \tag{12}$$

where $g(t)$, given by (9), is continuous and bounded by $|g(t)| \leqslant \mathrm{tr}(A(t)) = \mathrm{tr}(A(0))$. Under these conditions, the solution of (12) is

$$y(t) = y(0) \exp(G(t)),$$

where $G(t) = \int_0^t g(t)\,\mathrm{d}t$. Thus $y(t)$ retains its sign over $(-\infty, \infty)$, i.e., $y(t)$ has the same sign as $y(0)$. Now consider the various cases:

(a) $A(0)$ is STP. By continuity there is an interval (a, b) around 0 in which $A(t)$ is STP. Suppose if possible that, as t increases from zero, one or more minors of A first become zero at $t = b$. Then $A(b)$ is merely NTP. But $A(0)$ is STP, so that $c_p(0) > 0, \quad p = 1, 2, \dots, n$ and thus $c_p(b) > 0$. But then $A(b)$ is NTP and has strictly positive corner minors: $A(b)$ is STP. This contradiction implies $A(t)$ is STP for all $t > 0$; an exactly similar argument shows that $A(t)$ is STP for all $t < 0$. $A(t)$ is STP for all t.

(b) $A(0)$ is TP. For every $k = 1, 2, \ldots,$ $C(k, 0) = P(k)\{A(0) + \exp(-k)I\}P(k)$ is STP. Under the Toda flow (7), $C(k, t)$ remains STP. Now we apply standard results on the variation of solutions of o.d.e.'s with respect to initial conditions and parameters, e.g. [4, Theorem 4.1]. The Toda flow equation may be written

$$\dot{A} = f(A).$$

The matrix $f(A(t))$ is bounded. In the Frobenius norm

$$||f(A(t))||_2 \leqslant 2||A(t)||_2^2 = 2||A(0)||_2^2$$

so that

$$\lim_{k \longrightarrow \infty} C(k, 0) = A(0)$$

implies

$$\lim_{k \longrightarrow \infty} C(k, t) = A(t).$$

But $C(k, t)$ is STP so that $\lim_{k \longrightarrow \infty} C(k, t)$ is TP. Thus $A(t)$ is TP.

(c) $A(0)$ is NTP. $A(0)$ is TP, so that $A(t)$ is TP, $\det(A(t)) = \det(A(0)) \neq 0$, so that $A(t)$ is NTP.

(d) $A(0)$ is O. First $A(0)$ is NTP, so that $A(t)$ is NTP, and hence $B(t) = A^m(t)$ is NTP, for all $m = 1, 2, \ldots$ Secondly, for some m, with $1 \leqslant m \leqslant n - 1$, $B(0) = A^m(0)$ is STP so that, by (a), $B(t)$ is STP: $A(t)$ is O.

(e) $A(0)$ is SO; $\tilde{A}(0)$ is O; $\tilde{A}(t)$ is O; $A(t)$ is SO. $\square$

3. Concluding remarks

In this paper we have been concerned with signs, with patterns of signs which are preserved under Toda flow. There is a large body of research connected with matrix *shapes*, i.e., patterns of zero and non-zero terms, that are preserved under QR transformation [2] or under a wide class of Toda-like flows [3]. The basic pattern that is preserved, in both cases, is the *staircase*.

A sequence $p = \{p_1, p_2, \ldots, p_n\}$ is said to be a staircase sequence if $1 \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n \leqslant n$ and $p_i \geqslant i,\ i = 1, 2, \ldots, n$. A symmetric matrix is p-staircase if $a_{ij} = 0$ for $j > p_i,\ i = 1, 2, \ldots, n$. For a strict band matrix with half-band width r, $p_i = i + r$. In that case $p_1 < p_2 < \cdots < p_{n-r}$.

The Toda flow (7) has the property that if A is a symmetric p-staircase matrix, and a_{ij} lies outside the staircase, then $\dot{a}_{ij} = 0$. Thus, terms initially outside the staircase remain zero. It may easily be verified that if a_{ij} is on the tip of a stair of the staircase i.e., $j = p_i$ and either $i = 1$ or $p_i > p_{i-1}$, then $\dot{a}_{ij} = (a_{jj} - a_{ii})a_{ij}$, so that if $a_{ij}(0) >, =, < 0$ then $a_{ij}(t) >, =, < 0$, respectively. Other terms on the boundary of the staircase may become zero or change their signs. If $A(0)$ is a strict band matrix with half-band width r, so that all the terms in the outermost diagonal are non-zero, then since $p_1 < p_2 < \cdots < p_{n-r}$, all the terms on that diagonal retain their signs.

The definition of a staircase matrix does not state that $a_{ij} \neq 0$ for $i \leqslant j \leqslant p_i$; there may be zero terms, i.e., holes, in the staircase. For a general symmetric matrix under Toda flow, even if $A(0)$ is a p-staircase with no holes, then holes may appear in the p-staircase for $A(t)$.

What differences occur when $A(0)$ has some TP property? Markham [10] effectively showed that if $A \in S_n$ is O, then A is a p-staircase *with no holes*: all the terms inside and on the staircase are positive. It may be verified that this result still holds even if $A(0)$ is merely NTP. This means that if $A(0)$ is NTP, so that $A(0)$ is a p -staircase with no holes, then $A(t)$ will remain the same p-staircase pattern, and all the terms inside and on the staircase will remain positive.

We have deliberately phrased the problem and the results in the simplest form; undoubtedly they may be generalised in many ways, but we leave this to others.

References

[1] T. Ando, Totally positive matrices, Linear Algebra Appl. 90 (1987) 165–219.
[2] P. Arbenz, G. Golub, Matrix shapes invariant under the symmetric *QR* algorithm, Numer. Linear Algebra Appl. 2 (1995) 87–93.
[3] D.A. Ashlock, K.R. Driessel, I.R. Hentzel, On matrix structures invariant under Toda-like isospectral flows, Linear Algebra Appl. 254 (1997) 29–48.
[4] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, Krieger, Malabar, FL, 1955.
[5] M. Fekete, Über ein problem von laguerre, Rend. Circ. Math. Palermo 34 (1913) 89–100, 110–120.
[6] F.R. Gantmakher, M.G. Krein, Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems (English transl. 1961), Department of Commerce, Washington, 1950.
[7] M. Gasca, J.M. Peña, Total positivity and Neville elimination, Linear Algebra Appl. 165 (1992) 25–44.
[8] G.M.L. Gladwell, Inverse Problems in Vibration, Kluwer Academic Publishers, Dordrecht, 1986.
[9] G.M.L. Gladwell, Total positivity and the *QR* algorithm, Linear Algebra Appl. 271 (1998) 257–272.
[10] T. Markham, On oscillatory matrices, Linear Algebra Appl. 3 (1970) 143–158.