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Isospectral vibrating beams

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It has long been known that two scaling factors and three spectra, corresponding to three different end conditions, are required to determine the masses, lengths and stiffnesses in a discrete model of a beam in flexural vibration. What had not been known was how to find a family of beams that all have the same spectrum under one set of end conditions, say those corresponding to a cantilever. This paper presents two procedures for finding families of such isospectral beams. The first uses shifted QR factorization and yields a four-parameter family. The second uses Toda flow to find another, more restricted family.

Keywords: isospectral; vibration; beam; QR factorization; Toda flow

1. Introduction

This paper concerns the undamped free infinitesimal vibrations of a discrete elastic system $\mathcal{S}$ about an equilibrium configuration. The governing equation for such a system has the form

$$\boldsymbol{M}\ddot{q} + \boldsymbol{K}q = \boldsymbol{0}, \tag{1.1}$$

where $\boldsymbol{M}$, $\boldsymbol{K}$ are the mass and stiffness matrices of $\mathcal{S}$. The natural frequencies $(\omega_i)_1^n$ of $\mathcal{S}$ are related to the eigenvalues $(\lambda_i)_1^n$ of

$$(\boldsymbol{K} - \lambda\boldsymbol{M})\boldsymbol{q} = \boldsymbol{0} \tag{1.2}$$

by $\lambda_i = \omega_i^2$. The set $(\lambda)_1^n$ is called the *spectrum* of $\mathcal{S}$ and is denoted by $\sigma(\boldsymbol{M}, \boldsymbol{K})$. Two systems $\mathcal{S}$ and $\mathcal{S}'$ are said to be *isospectral* if

$$\sigma(\boldsymbol{M}', \boldsymbol{K}') = \sigma(\boldsymbol{M}, \boldsymbol{K}). \tag{1.3}$$

In general, the most that can be stated about $\boldsymbol{M}$ and $\boldsymbol{K}$ is that they are symmetric; $\boldsymbol{M}$ is positive-definite (PD), $\boldsymbol{K}$ is PD if $\mathcal{S}$ is anchored and positive-semi-definite (PSD) otherwise. These conditions ensure that the spectrum $\sigma(\boldsymbol{M}, \boldsymbol{K})$ is non-negative, and positive if $\mathcal{S}$ is anchored.

Since $\boldsymbol{M}$ is PD, it may be factorized in the form $\boldsymbol{M} = \boldsymbol{L}\boldsymbol{L}^{\mathrm{T}}$, where $\boldsymbol{L}$ is lower triangular, and (1.2) may be reduced to standard form,

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{u} = \boldsymbol{0}, \tag{1.4}$$

where $\boldsymbol{A} = \boldsymbol{L}^{-1}\boldsymbol{K}\boldsymbol{L}^{-\mathrm{T}}$ is PSD and $\boldsymbol{L}^{\mathrm{T}}\boldsymbol{q} = \boldsymbol{u}$. The matrix $\boldsymbol{A}$ is called the *mass-reduced stiffness matrix* of $\mathcal{S}$; the spectrum of $\mathcal{S}$ is denoted simply by $\sigma(\boldsymbol{A})$. It is

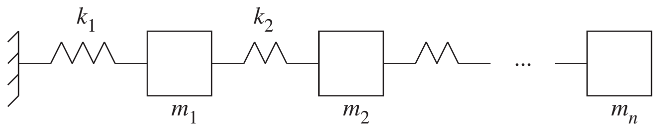

Figure 1. An in-line mass–spring system.

well known that two symmetric matrices $\boldsymbol{A}$, $\boldsymbol{A}'$ are isospectral, $\sigma(\boldsymbol{A}') = \sigma(\boldsymbol{A})$, if and only if

$$\boldsymbol{A}' = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{Q}, \tag{1.5}$$

where $\boldsymbol{Q}$ is an orthogonal matrix, i.e. $\boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} = \boldsymbol{I}$. This means that, at the level of mass-reduced stiffness matrices, the search for isospectral systems is straightforward: two systems $\mathcal{S}$ and $\mathcal{S}'$ are isospectral if and only if their mass-reduced stiffness matrices are linked by (1.5), for some orthogonal matrix $\boldsymbol{Q}$.

However, this does not end the search for isospectral systems at the physical (mechanical) level. There the search is for mass and stiffness matrices that have a *specified form* defined by the connectivity of $\mathcal{S}$. This connectivity is mirrored in the pattern of zero and non-zero elements of $\boldsymbol{K}$ and/or $\boldsymbol{M}$. There may also be specified *sign patterns* for the elements of $\boldsymbol{K}$ and/or $\boldsymbol{M}$. If $\boldsymbol{K}$, $\boldsymbol{M}$ have such specified forms, $\boldsymbol{A}$ is constructed as $\boldsymbol{L}^{-1}\boldsymbol{K}\boldsymbol{L}^{-\mathrm{T}}$ and $\boldsymbol{Q}$ is an arbitrary orthogonal matrix, then there is no guarantee that $\boldsymbol{A}'$, constructed as in (1.5), may be written as $\boldsymbol{A}' = (\boldsymbol{L}')^{-1}\boldsymbol{K}'(\boldsymbol{L}')^{-\mathrm{T}}$, where $\boldsymbol{K}'$ and $\boldsymbol{M}' = \boldsymbol{L}'\boldsymbol{L}'^{\mathrm{T}}$ also have the specified form. This is the crux of the problem: find those orthogonal matrices $\boldsymbol{Q}$ that guarantee such an invariance of form. This paper achieves this aim for the particular case of a vibrating beam in flexure. First, however, we consider the comparatively simple case of an in-line mass–spring system.

2. A simple example

The simplest example of an n-degree-of-freedom system is an in-line system of n masses $(m_i)_1^n$ connected by springs $(k_i)_1^n$, as shown in figure 1, for which

$$\boldsymbol{K} = \boldsymbol{E}\hat{\boldsymbol{K}}\boldsymbol{E}^{\mathrm{T}}, \qquad \boldsymbol{M} = \mathrm{diag}(m_1, m_2, \ldots, m_n), \tag{2.1}$$

where $\hat{\boldsymbol{K}} = \mathrm{diag}(k_1, k_2, \ldots, k_n)$ and $\boldsymbol{E}$ is the difference matrix with inverse $\boldsymbol{E}^{-1}$, given by

$$\boldsymbol{E} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{bmatrix} \qquad \boldsymbol{E}^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{2.2}$$

This model is mathematically equivalent to a lumped-mass finite-element model of a rod in longitudinal vibration, to a set of point masses vibrating transversely on a taut string, and to a finite-difference or finite-element approximation to a Sturm–Liouville problem.

The properties of this system have been studied exhaustively, but since we will make use of known results for this simple system in studying the problem of finding isospectral beams in flexural vibration, we recall some of these known results.

The problem has a long history (for which, see Gantmakher & Krein 1950; Gladwell 1986*a*). It is known that $\boldsymbol{M}$, $\boldsymbol{K}$ may be reconstructed, apart from a single scaling factor, from two spectra: $(\lambda_i)_1^n$ for the system shown in figure 1 and $(\mu_i)_1^{n-1}$ for the system with the mass m_n fixed. The two spectra must interlace (Gladwell 1985), so that

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n. \tag{2.3}$$

Thus, given one system with spectra $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$, we may find all other isospectral in-line mass–spring systems simply by choosing another spectrum $(\mu_i')_1^n$ satisfying

$$0 < \lambda_1 < \mu_1' < \lambda_2 < \cdots < \mu_{n-1}' < \lambda_n.$$

It is known that reconstruction from two spectra $(\lambda)_1^n$ and $(\mu_i)_1^{n-1}$ is equivalent to reconstructing the system from one spectrum $(\lambda_i)_1^n$ and the set $(u_n^{(i)})_1^n$ of last components of the normalized eigenvectors of (1.2). These components satisfy

$$\sum_{i=1}^{n}(u_n^{(i)})^2 = \frac{1}{m_n}.$$

Each member of the isospectral family of systems with eigenvalues $(\lambda_i)_1^n$ thus corresponds to a point of the strictly positive orthant of the n-sphere

$$\sum_{i=1}^{n}(u_n^{(i)})^2 = \frac{1}{m_n}.$$

These two ways of reconstructing new systems isospectral to a given one use the *spectrum* of the original system. There is another way that starts from the mass-reduced stiffness matrix $\boldsymbol{A}$, constructs a new isospectral mass-reduced stiffness matrix $\boldsymbol{A}'$ and then reconstructs $\boldsymbol{K}'$, $\boldsymbol{M}'$ from it.

For $\mathcal{S}$ in figure 1, $\boldsymbol{M}$ is diagonal. Factorize it as $\boldsymbol{M} = \boldsymbol{D}^2$, $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. Then

$$\boldsymbol{A} = \boldsymbol{D}^{-1}\boldsymbol{K}\boldsymbol{D}^{-1} \tag{2.4}$$

is a symmetric *Jacobi* matrix of the form

$$\boldsymbol{A} = \begin{bmatrix} a_1 & -b_1 & & & \\ -b_1 & a_2 & -b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -b_{n-1} \\ & & & -b_{n-1} & a_n \end{bmatrix}. \tag{2.5}$$

Gladwell (1995) showed that if μ is not an eigenvalue of $\boldsymbol{A}$, and if $\boldsymbol{A}-\mu\boldsymbol{I}$ is factorized, as it always can be, in the form

$$\boldsymbol{A} - \mu\boldsymbol{I} = \boldsymbol{Q}\boldsymbol{R}, \tag{2.6}$$

where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{R}$ if upper triangular with *positive* diagonal, then the matrix $\boldsymbol{A}'$, defined by

$$\boldsymbol{A}' - \mu\boldsymbol{I} = \boldsymbol{R}\boldsymbol{Q}, \tag{2.7}$$

is another symmetric Jacobi matrix, with negative co-diagonal, which is isospectral to $\boldsymbol{A}$,

$$\boldsymbol{A}' = \mu\boldsymbol{I} + \boldsymbol{R}\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}}(\mu\boldsymbol{I} + \boldsymbol{Q}\boldsymbol{R})\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{Q}. \tag{2.8}$$

To complete the solution of the problem, we must factorize the new $\boldsymbol{A}'$ in the form

$$\boldsymbol{A}' = (\boldsymbol{D}')^{-1}\boldsymbol{K}'(\boldsymbol{D}')^{-1}, \tag{2.9}$$

where $\boldsymbol{K}' = \boldsymbol{E}\hat{\boldsymbol{K}}'\boldsymbol{E}^{\mathrm{T}}$. To effect this factorization, we note that a mass–spring system, of the form shown in figure 1, has the property that a single static force k_1', applied to the first mass, will shift all the masses by one unit to the right,

$$\boldsymbol{K}'\{1, 1, \dots, 1\} = \{k_1', 0, \dots, 0\}. \tag{2.10}$$

But, from (2.9), $\boldsymbol{K}' = \boldsymbol{D}'\boldsymbol{A}'\boldsymbol{D}'$, so that

$$\boldsymbol{D}'\boldsymbol{A}\boldsymbol{D}'\{1, 1, \dots, 1\} = \boldsymbol{D}'\boldsymbol{A}'\{d_1', d_2', \dots, d_n'\} = \{k_1', 0, \dots, 0\},$$

i.e.

$$\boldsymbol{A}'\{d_1', d_2', \dots, d_n'\} = \{f_1', 0, \dots, 0\}, \tag{2.11}$$

where $f_1' = k_1'/d_1'$. This yields the required reconstruction.

(i) Take $\boldsymbol{K}$, $\boldsymbol{M}$ given by (2.1) and construct $\boldsymbol{A}$ from (2.4).

(ii) Choose μ, not an eigenvalue of $\boldsymbol{A}$, and factorize $\boldsymbol{A} - \mu\boldsymbol{I} = \boldsymbol{Q}\boldsymbol{R}$. Note that, since the spectrum of $\boldsymbol{A}$ consists of a finite set $(\lambda_i)_1^n$, *almost any* μ will *not* be an eigenvalue of $\boldsymbol{A}$.

(iii) Form $\boldsymbol{A}' = \mu\boldsymbol{I} + \boldsymbol{R}\boldsymbol{Q}$.

(iv) Take $f_1' = 1$ and solve (2.11) for $\boldsymbol{d}'$. Because $\boldsymbol{A}'$ is a Jacobi matrix, its inverse is positive (see the remarks in § 4); $\boldsymbol{d}'$ is positive.

(v) Put $\boldsymbol{K}' = \boldsymbol{D}'\boldsymbol{A}'\boldsymbol{D}'$, where $\boldsymbol{D}' = \mathrm{diag}(d_1', d_2', \dots, d_n')$.

(vi) The matrix $\hat{\boldsymbol{K}}' = \boldsymbol{E}^{-1}\boldsymbol{K}'\boldsymbol{E}^{-\mathrm{T}} = \boldsymbol{E}^{-1}\boldsymbol{D}'\boldsymbol{A}'\boldsymbol{D}'\boldsymbol{E}^{-\mathrm{T}}$ is diagonal and contains the stiffnesses $(k_i')_1^n$.

The only step we have not discussed is (vi). Use the notation (2.5), with primes, for $\boldsymbol{A}'$, and consider the matrix product

$$\boldsymbol{A}'\boldsymbol{D}'\boldsymbol{E}^{-\mathrm{T}} = \boldsymbol{A}'\begin{bmatrix} d_1' & & & \\ d_2' & d_2' & & \\ \vdots & \vdots & \ddots & \\ d_n' & d_n' & & d_n' \end{bmatrix} = \begin{bmatrix} 1 & -b_1'd_2' & & \\ & b_1'd_1' & \ddots & \\ & & \ddots & -b_{n-1}'d_n' \\ & & & b_{n-1}'d_{n-1}' \end{bmatrix}. \tag{2.12}$$

It is zero below the diagonal. Since $\boldsymbol{E}^{-1}$ is zero below the diagonal, then so must $(\boldsymbol{E}^{-1}\boldsymbol{D}')(\boldsymbol{A}'\boldsymbol{D}'\boldsymbol{E}^{-\mathrm{T}})$ be; but the latter is symmetric, it is diagonal, it is $\hat{\boldsymbol{K}}'$,

$$\hat{\boldsymbol{K}}' = \mathrm{diag}(d_1', b_1'd_1'd_2', \dots, b_{n-1}'d_{n-1}'d_n'). \tag{2.13}$$

Thus

$$\boldsymbol{E}^{-1}\boldsymbol{D}'\boldsymbol{A}'\boldsymbol{D}'\boldsymbol{E}^{-\mathrm{T}} = \hat{\boldsymbol{K}}', \tag{2.14}$$

so that

$$\boldsymbol{A}' = (\boldsymbol{D}')^{-1}\boldsymbol{E}\hat{\boldsymbol{K}}'\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{D}')^{-1}, \tag{2.15}$$

so that if

$$\boldsymbol{K}' = \boldsymbol{E}\hat{\boldsymbol{K}}'\boldsymbol{E}^{\mathrm{T}}, \qquad \boldsymbol{M}' = (\boldsymbol{D}')^2, \tag{2.16}$$

then

$$\sigma(\boldsymbol{M}', \boldsymbol{K}') = \sigma(\boldsymbol{M}, \boldsymbol{K}); \tag{2.17}$$

$\mathcal{S}'$ is isospectral to $\mathcal{S}$.

Note that the factorization of $\boldsymbol{A}'$ is unique, apart from a single scaling factor;

$$\boldsymbol{D}'' = \alpha\boldsymbol{D}', \qquad \boldsymbol{M}'' = \alpha^2\boldsymbol{M}', \qquad \hat{\boldsymbol{K}}'' = \alpha^2\hat{\boldsymbol{K}}' \tag{2.18}$$

provides another isospectral system $\mathcal{S}''$.

This analysis shows that *any* symmetric tridiagonal PD matrix with strictly negative codiagonal (i.e. any Jacobi matrix) may be factorized as in (2.15). We will use this result later.

3. History of the problem

A beam vibrating flexurally may be modelled as a continuum or as a discrete system; the governing equations are, respectively, a fourth-order differential equation and a pentadiagonal matrix equation.

Free vibration problems may be divided into *direct* and *inverse* problems. The former concern the determination of natural frequencies of a given system, the latter concern the reconstruction of a system with given natural frequencies. Direct problems for a vibrating beam, modelled as a continuum or as a discrete system, are well understood and offer few new challenges. Inverse problems for continuum or discrete models of a beam are now finally understood, but they are, by nature, much more complicated than the corresponding problems for continuum or discrete models of a vibrating rod. It is necessary to have some understanding of the inverse problems in order to be able to construct isospectral families of beams.

It was stated earlier, in § 2, that a discrete model of a vibrating *rod* may be reconstructed from two spectra, $(\lambda_i)_1^n$ for the fixed–free-end conditions, $(\mu_i)_1^{n-1}$ for the fixed–fixed conditions. They must interlace according to (2.3). If they do, they yield the end components $(u_n^{(i)})_1^n$ of the eigenvectors and, from these components, $\boldsymbol{A}$, and hence $\boldsymbol{M}$, $\boldsymbol{K}$, may be reconstructed.

Barcilon (1976, 1982) was the first to discuss the corresponding problem for a discrete model of a beam. His analysis was incomplete. Gladwell (1984) considered the model in figure 2, and it is this model that will be discussed here. It consists of n masses $(m_i)_1^n$, joined by rigid rods of lengths $(l_i)_1^n$, connected by torsional springs of stiffness $(k_i)_1^n$.

For the system in figure 1, two spectra were needed for reconstruction. For the system in figure 2, three spectra are needed: $(\lambda_i)_1^n$, $(\sigma_i)_1^{n-1}$ and $(\mu_i)_1^{n-1}$, corresponding, respectively, to free, sliding and pinned conditions at the right-hand end. Gladwell

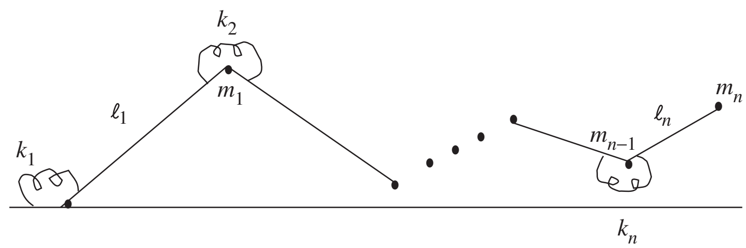

Figure 2. A discrete model of a vibrating beam.

(1985) showed that these must interlace according to the inequalities

$$0 < \lambda_i < \sigma_1 < \mu_1 < \lambda_2 < \sigma_2 < \mu_2 < \cdots < \lambda_n. \tag{3.1}$$

As for the rod, these spectra yield the end displacements $(u_n^{(i)})_1^n$ and the end slopes $(\theta_n^{(i)})_1^n$ of the eigenmodes of the fixed–free beam, and from these displacements and slopes the masses, lengths and stiffnesses may be reconstructed.

However, unlike the rod, mere interlacing, as in (3.1), is insufficient to ensure that the reconstruction analysis will yield sensible, i.e. positive, values for the lengths $(l_i)_1^n$ entering the model. It is necessary (and sufficient) that the displacements $u_n^{(i)}$ and slopes $\theta_n^{(i)}$ satisfy a complicated interlocking set of inequalities, as given in Gladwell (1984). This means that, unlike the rod, we cannot seek a family of isospectral beams by choosing new spectra $(\sigma_i')_1^{n-1}$, $(\mu_i')_1^{n-1}$ interlacing the $(\lambda_i)_1^n$ in the sense

$$0 < \lambda_i < \sigma_1' < \mu_1' < \lambda_2 < \sigma_2' < \mu_2' < \cdots < \lambda_n. \tag{3.2}$$

Instead, we will use a procedure based on isospectral mass-reduced stiffness matrices. Before proceeding to this analysis, we note that the solution of inverse problems for a continuum model was initiated by McLaughlin (1984) and completed by Gladwell (1986*b*); the solution follows closely along the lines of the corresponding solution for the discrete model. Gottlieb (1987) obtained some families of isospectral continuum beams, and this method was generalized by Subramanian & Raman (1996), but the quest for more general families of isospectral beams remains an open problem.

4. Structure and signs

As stated earlier, the search for isospectral systems involves finding a system $\mathcal{S}'$ with mass and stiffness matrices $\boldsymbol{M}'$, $\boldsymbol{K}'$ having specified forms and which satisfy (1.3) for some given $\boldsymbol{M}$, $\boldsymbol{K}$. It is shown in Gladwell (1984, 1986*a*) that, for the cantilever beam of figure 2,

$$\boldsymbol{K} = \boldsymbol{E}\boldsymbol{L}^{-1}\boldsymbol{E}\hat{\boldsymbol{K}}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{L}^{-1}\boldsymbol{E}^{\mathrm{T}}, \qquad \boldsymbol{M} = \boldsymbol{D}^2, \tag{4.1}$$

where $\boldsymbol{E}$ is again given by (2.2), $\boldsymbol{L} = \mathrm{diag}(l_1, l_2, \ldots, l_n)$, $\hat{\boldsymbol{K}} = \mathrm{diag}(k_1, k_2, \ldots, k_n)$, $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. The mass-reduced stiffness matrix

$$\boldsymbol{A} = \boldsymbol{D}^{-1}\boldsymbol{K}\boldsymbol{D}^{-1} \tag{4.2}$$

is thus pentadiagonal with alternate signs, as in

$$\boldsymbol{A}=\begin{bmatrix} a_1 & -b_1 & c_1 & & & & \\ -b_1 & a_2 & -b_2 & c_2 & & & \\ c_1 & -b_2 & a_3 & -b_3 & c_3 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c_{n-2} \\ & & & \ddots & \ddots & \ddots & -b_{n-1} \\ & & & & c_{n-2} & -b_{n-1} & a_n \end{bmatrix}. \tag{4.3}$$

Any isospectral system $\mathcal{S}'$ must have a mass-reduced stiffness matrix $\boldsymbol{A}'$ with this same form. However, the restrictions do not merely relate to the signs of *elements* of $\boldsymbol{A}$ and $\boldsymbol{A}'$; they relate to the signs of all the non-zero *minors* of $\boldsymbol{A}$ and $\boldsymbol{A}'$. We recall that a minor of order p of a matrix is the determinant of the submatrix consisting of elements in p chosen rows $i_1, i_2, \dots, i_p$ and p chosen columns $j_1, j_2, \dots, j_p$, and is denoted by $\boldsymbol{A}(i_1, i_2, \dots, i_p; j_1, j_2, \dots, j_p)$. It is necessary to introduce the concept of *total positivity* and some related concepts.

Following Karlin (1968), we say that a matrix $\boldsymbol{B}$ is

(i) totally positive (TP) if *all* its minors are non-negative;

(ii) *non-singular* (NTP) if it is non-singular and TP;

(iii) strictly TP (STP) if all minors are strictly positive;

(iv) oscillatory (O) if $\boldsymbol{B}$ is TP and $\boldsymbol{B}^m$ is STP for some positive integer m. It is known that $\boldsymbol{B}$ is O if and only if $\boldsymbol{B}$ is NTP and $b_{i,i+1} > 0$, $b_{i+1,i} > 0$ for $i = 1, 2, \dots, n-1$;

(v) sign-oscillatory (SO) if $\tilde{\boldsymbol{B}} = \boldsymbol{ZBZ}$ is O, where $\boldsymbol{Z} = \operatorname{diag}(1, -1, 1, \cdots \pm 1)$.

The concepts O and SO are linked by the fundamental result

$$\boldsymbol{A} \text{ is SO if and only if } \boldsymbol{B} = \boldsymbol{A}^{-1} \text{ is O.}$$

Note that the concepts TP, NTP, STP, O and SO relate to an arbitrary real square matrix, not just to symmetric matrices, although the matrices encountered in this paper are all symmetric. One of the fundamental results of total positivity theory is that the eigenvalues of an oscillatory matrix (whether symmetric or not) are real, positive and distinct.

Using these concepts, we may classify the mass-reduced stiffness matrices of the rod (2.5) and beam (4.1); they are SO. This may readily be verified by carrying out the matrix multiplication involved in constructing the $\boldsymbol{A}$.

Since $\boldsymbol{Z}$ is orthogonal, ($\boldsymbol{ZZ}^{\mathrm{T}} = \boldsymbol{Z}^2 = \boldsymbol{I}$), $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}} = \boldsymbol{ZAZ}$ have the same eigenvalues. This means that, if $\boldsymbol{A}$ is SO, its spectrum is positive and distinct; the $\boldsymbol{A}$ for a rod and a beam are SO, so their spectra are positive and distinct, as is known.

It may easily be shown that a symmetric PD tridiagonal matrix with strictly negative codiagonal, i.e. a Jacobi matrix, is SO. Its inverse is therefore O, and it is, in fact, a full matrix with strictly positive elements (in the notation of Berman

& Plemmons (1994), $\boldsymbol{A}$ is an $\boldsymbol{M}$-matrix); this was the result used to deduce that the d_i', obtained by solving (2.11), were strictly positive.

The first step in finding a discrete-model cantilever beam isospectral to a given one is to find a symmetric pentadiagonal SO matrix $\boldsymbol{A}'$, isospectral to a given symmetric pentadiagonal SO matrix $\boldsymbol{A}$. To do this, we use a generalization of the procedure used for generating a new Jacobi matrix $\boldsymbol{A}'$ from a given Jacobi matrix $\boldsymbol{A}$, by using shifted QR factorization.

Gladwell (1998) proved a very general result, which includes the following. If $\boldsymbol{A}$ is symmetric and O, μ is not an eigenvalue of $\boldsymbol{A}$, $\boldsymbol{A}-\mu\boldsymbol{I}=\boldsymbol{QR}$, where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{R}$ is upper triangular with positive diagonal, then $\boldsymbol{A}'$, given by $\boldsymbol{A}'-\mu\boldsymbol{I}=\boldsymbol{RQ}$ is symmetric and O. We note that if $\boldsymbol{A}$ is SO, rather than O, then $\boldsymbol{A}'$ is SO also. (For, if $\boldsymbol{A}$ is SO and $\boldsymbol{A}-\mu\boldsymbol{I}=\boldsymbol{QR}$, then $\tilde{\boldsymbol{A}}$ is O and $\tilde{\boldsymbol{A}}-\mu\boldsymbol{I}=\boldsymbol{ZQZ}\cdot\boldsymbol{ZRZ}=\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{R}}$. Thus $\tilde{\boldsymbol{A}}'$, given by $\tilde{\boldsymbol{A}}'-\mu\boldsymbol{I}=\tilde{\boldsymbol{R}}\tilde{Q}$, is O. Therefore, $\boldsymbol{A}'$ is SO.)

Note that (2.8) still holds; $\sigma(\boldsymbol{A}')=\sigma(\boldsymbol{A})$. The matrix $\boldsymbol{A}'$ is obtained from $\boldsymbol{A}$ by shifted $\boldsymbol{QR}$ factorization and reversal of the factors, and it may easily be verified that the procedure by which $\mathbf{A}'$ is obtained from $\boldsymbol{A}$ maintains bandwidth; if $\boldsymbol{A}$ is pentadiagonal, so is $\boldsymbol{A}'$. We can prove more. If $\boldsymbol{A}$ is obtained from (4.1), (4.2), so that $c_i>0$, $i=1,2,\ldots,n-2$, then $\boldsymbol{A}'$, written as (4.2) with primes, has strictly positive outer diagonal, i.e. $c_i'>0$. To prove this, we obtain another equation linking $\boldsymbol{A}$ and $\boldsymbol{A}'$,

$$\boldsymbol{RA}=\mu\boldsymbol{R}+\boldsymbol{RQR}=(\mu\boldsymbol{I}+\boldsymbol{RQ})\boldsymbol{R}=\boldsymbol{A}'\boldsymbol{R}. \tag{4.4}$$

Equating the i, $i+2$ terms on both sides gives

$$r_{ii}c_i=c_i'r_{i+2,i+2}. \tag{4.5}$$

Since the diagonal rs are strictly positive, $c_i>0$ implies $c_i'>0$.

The procedure described below requires one more result. The inverse $(\boldsymbol{A}')^{-1}=\boldsymbol{B}'$, like $\boldsymbol{A}^{-1}=\boldsymbol{B}$ obtained from (4.1), (4.2), is a strictly positive matrix, i.e. all its elements are positive, just like the inverse of the Jacobi matrix $\boldsymbol{A}'$ in § 2. We know that, since $\boldsymbol{A}'$ is SO, $\boldsymbol{B}'$ is O; this means that its elements are non-negative. We prove that they are, in fact, all strictly positive. First we prove that $b_{n1}'>0$. Taking inverses of the two sides of (4.4), we find

$$\boldsymbol{BR}^{-1}=\boldsymbol{R}^{-1}\boldsymbol{B}', \tag{4.6}$$

and equating the n, 1 terms in the products gives

$$b_{n1}r_{11}^{-1}=r_{nn}^{-1}b_{n1}', \tag{4.7}$$

so that $b_{n1}'>0$, because $b_{n1}>0$. Markham (1970) showed that if $\boldsymbol{B}'$ is O, then $b_{n1}'>0$ implies that $\boldsymbol{B}'$ is strictly positive. For, since $\boldsymbol{B}'$ is O,

$$\begin{vmatrix} b_{i1}' & b_{ii}' \\ b_{n1}' & b_{ni}' \end{vmatrix} \geqslant 0, \quad i=2,3,\ldots,n-1, \tag{4.8}$$

but $b_{n1}'>0$, $b_{ii}'>0$, $b_{i1}'\geqslant 0$, $b_{ni}'\geqslant 0$ imply $b_{i1}'>0$, $b_{ni}'>0$. Thus the first column and last row of $\boldsymbol{B}'$ is positive. Now consider

$$\begin{vmatrix} b_{ij}' & b_{ii}' \\ b_{nj}' & b_{ni}' \end{vmatrix} \geqslant 0, \quad j<i<n. \tag{4.9}$$

Again, this implies $b_{ij}'>0$.

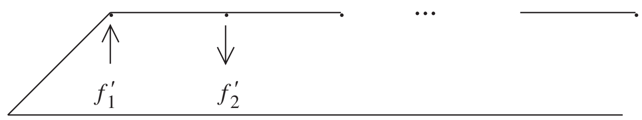

Figure 3. Two static forces applied to the beam to produce unit displacements.

5. Reconstruction of an isospectral beam

The procedure initially follows the lines of that used in § 2.

(i) Take $\boldsymbol{K}$, $\boldsymbol{M}$ given by (4.1) and construct $\boldsymbol{A}$ from (4.2).

(ii) Choose μ not an eigenvalue of $\boldsymbol{A}$ and factorize $\boldsymbol{A} - \mu\boldsymbol{I} = \boldsymbol{QR}$.

(iii) Form $\boldsymbol{A}' = \mu\boldsymbol{I} + \boldsymbol{RQ}$.

Now it remains to factorize $\boldsymbol{A}'$. To do so, we use the static behaviour of the beam model. To produce unit transverse displacements of the masses, we must apply a force f'_1 to mass m'_1 and a force $-f'_2$ to mass m'_2, as in figure 3,

$$\boldsymbol{K}'\{1, 1, \ldots, 1\} = \{f'_1, -f'_2, 0, \ldots, 0\}. \tag{5.1}$$

But $\boldsymbol{K}' = \boldsymbol{D}'\boldsymbol{A}'\boldsymbol{D}'$, so that, as before,

$$\boldsymbol{A}'\{d'_1, d'_2, \ldots, d'_n\} = \{g'_1, -g'_2, 0, \ldots, 0\}, \tag{5.2}$$

where $g'_i = f'_i/d'_i$, $i = 1, 2$. The solution of (5.2) is

$$d'_i = b'_{i1}g'_1 - b'_{i2}g'_2, \quad i = 1, 2, \ldots, n, \tag{5.3}$$

where $\boldsymbol{B}' = (\boldsymbol{A}')^{-1}$. Choose $g'_1 = 1$ and take g'_2, so that $d'_n > 0$, i.e.

$$0 < g'_2 < \frac{b'_{n1}}{b'_{n2}}. \tag{5.4}$$

(Recall that b'_{n1}, b'_{n2} are both strictly positive.) Now

$$d'_i > b'_{i1} - \frac{b'_{i2}b'_{n1}}{b'_{n2}} > \frac{b'_{i1}b'_{n2} - b'_{i2}b'_{n1}}{b'_{n2}} \geqslant 0, \tag{5.5}$$

because $\boldsymbol{B}'$ is O. Thus all the d'_i are strictly positive. Now we proceed as in § 2 and construct

$$\boldsymbol{C}' = \boldsymbol{E}^{-1}\boldsymbol{D}'\boldsymbol{A}'\boldsymbol{D}'\boldsymbol{E}^{\mathrm{T}}. \tag{5.6}$$

It is a Jacobi matrix with codiagonal

$$(-g'_2d'_2, -c'_1d'_1d'_3, \ldots, -c'_{n-2}d'_{n-2}d'_n).$$

Since $g'_2 > 0$ and $c'_i > 0$, $i = 1, \ldots, n-2$, this codiagonal is strictly negative. $\boldsymbol{C}'$ is PD because

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{C}'\boldsymbol{x} = (\boldsymbol{x}^{\mathrm{T}}\boldsymbol{E}^{-1}\boldsymbol{D}')\boldsymbol{A}'(\boldsymbol{D}'\boldsymbol{E}^{-\mathrm{T}}\boldsymbol{x}) = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}'\boldsymbol{y} > 0, \tag{5.7}$$

because $\boldsymbol{A}'$, being SO, is PD.

But we showed in § 2 that such a Jacobi matrix may be factorized as in (2.14). Therefore, replacing $\boldsymbol{D}'$ in (2.15) by $\boldsymbol{L}'$, respectively, we have

$$\boldsymbol{C}' = (\boldsymbol{L}')^{-1}\boldsymbol{E}\hat{\boldsymbol{K}}'\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{L}')^{-1}, \tag{5.8}$$

which, when combined with (5.6), yields

$$\boldsymbol{A}' = (\boldsymbol{D}')^{-1}\boldsymbol{E}(\boldsymbol{L}')^{-1}\boldsymbol{E}\hat{\boldsymbol{K}}'\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{L}')^{-1}\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{D}')^{-1}, \tag{5.9}$$

as required. Now, putting

$$\boldsymbol{M}' = (\boldsymbol{D}')^2, \boldsymbol{K}' = \boldsymbol{E}(\boldsymbol{L}')^{-1}\boldsymbol{E}\hat{\boldsymbol{K}}'\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{L}')^{-1}\boldsymbol{E}^{\mathrm{T}}, \tag{5.10}$$

we have

$$\sigma(\boldsymbol{M}', \boldsymbol{K}') = \sigma(\boldsymbol{M}, \boldsymbol{K}). \tag{5.11}$$

The family has two independent parameters, μ and g_2', and there are two scaling factors α, β. If

$$\boldsymbol{M}'' = \alpha^2\boldsymbol{M}', \qquad \hat{\boldsymbol{K}}'' = \beta^2\hat{\boldsymbol{K}}', \qquad \boldsymbol{L}'' = \beta\boldsymbol{L}'/\alpha, \tag{5.12}$$

then $\sigma(\boldsymbol{M}'', \boldsymbol{K}'') = \sigma(\boldsymbol{M}', \boldsymbol{K}')$.

6. Isospectral flow

There is another way to obtain a family of isospectral mass-reduced stiffness matrices: by setting up an isospectral flow equation. It is well known (Nanda 1985) that if $\boldsymbol{A}$ is symmetric and $\boldsymbol{S} = \boldsymbol{A}^{+\mathrm{T}} - \boldsymbol{A}^{+}$, where $\boldsymbol{A}^{+}$ is the upper triangle of $\boldsymbol{A}$, then the differential equation

$$\dot{\boldsymbol{A}} = \boldsymbol{A}\boldsymbol{S} - \boldsymbol{S}\boldsymbol{A} \equiv [\boldsymbol{A}, \boldsymbol{S}] \tag{6.1}$$

generates a matrix $\boldsymbol{A}(t)$, which is isospectral to $\boldsymbol{A}(0)$. It is known that (6.1), called the Toda-flow equation, preserves bandwidth, so that, in particular, if $\boldsymbol{A}(0)$ is pentadiagonal, then $\boldsymbol{A}(t)$ is pentadiagonal. Recently, Gladwell (2002) proved that (6.1) maintains all of the total positivity properties TP, NTP, STP, O and SO. Since this important result is not widely known, we repeat the essence of the analysis here. Theorem 2 of Gladwell (2002) states that if $\boldsymbol{A}(0)$ has one of the properties TP, NTP, STP, O and SO, then $\boldsymbol{A}(t)$ has the same property for all t. The argument runs as follows. First, suppose that $\boldsymbol{A}(0)$ is STP. This means that all the minors of $\boldsymbol{A}(0)$ are strictly positive. By continuity, all the minors of $\boldsymbol{A}(t)$ will be strictly positive in some open interval around 0, say (a, b). Suppose, if possible, that one or more of the minors became zero at $t = b$. Then $\boldsymbol{A}(b)$ would still be TP. Gladwell (1998) showed that if a symmetric matrix $\boldsymbol{A}$ is TP and the bottom-left *corner* minors

$$d_p = A[1, 2, \ldots, p \mid n-p+1, \ldots, n], \quad p = 1, 2, \ldots, n, \tag{6.2}$$

are *positive*, then $\boldsymbol{A}$ is STP. When the matrix $\boldsymbol{A}(t)$ satisfies (6.1), the corner minors satisfy the equation

$$\dot{d}_p(t) = g(t)d_p(t), \tag{6.3}$$

where

$$g(t) = \sum_{j=n-p+1}^{n} a_{jj}(t) - \sum_{j=1}^{p} a_{jj}(t).$$

The general solution of (6.3) is

$$d_p(t) = d_p(0)\exp(G(t)),$$

where

$$G(t) = \int_0^t g(t)\,\mathrm{d}t.$$

Thus, provided that $g(t)$ is bounded, and it is, since

$$|g(t)| \leqslant \mathrm{Tr}(\boldsymbol{A}(t)) = \mathrm{Tr}(\boldsymbol{A}(0)),$$

$d_p(t)$ maintains the same sign as $d_p(0)$, namely positive. We conclude that, since $\boldsymbol{A}(b)$ is TP and the corner minors are positive, $\boldsymbol{A}(b)$ is, in fact, STP, contrary to the hypothesis that one or more of the minors were zero when $t = b$. This contradiction implies that if $\boldsymbol{A}(0)$ is STP, then $\boldsymbol{A}(t)$ is STP for all t.

To show that if $\boldsymbol{A}(0)$ is simply TP, then $\boldsymbol{A}(t)$ is TP, we use the fact, due to Ando (1987), that a TP matrix may be approximated arbitrarily closely by an STP matrix. To extend the result to include matrices that are oscillatory, we use the fact that if $\boldsymbol{A}(t)$ satisfies (6.1), then $\boldsymbol{C}(t) = (\boldsymbol{A}(t))^m$ satisfies the same equation, i.e.

$$\dot{\boldsymbol{C}} = \boldsymbol{CS} - \boldsymbol{SC},$$

where $\boldsymbol{S} = \boldsymbol{A}^{+\mathrm{T}} - \boldsymbol{A}^{+}$, as before. Moreover, the corner minors of $\boldsymbol{C}(t)$ satisfy the same equation (6.3), and so maintain their sign. Thus, if $\boldsymbol{A}(0)$ is oscillatory, i.e. O, then, on the one hand, $\boldsymbol{A}(0)$ is TP, and hence $\boldsymbol{A}(t)$ is TP. On the other hand, $\boldsymbol{C}(0) = (\boldsymbol{A}(0))^m$ is STP for some $m \leqslant n-1$, so that, by the previous argument, $\boldsymbol{C}(t) = (\boldsymbol{A}(t))^m$ is STP, and hence $\boldsymbol{A}(t)$ is O.

If $\boldsymbol{A}(0)$ is SO, then $Z\boldsymbol{A}(0)Z$ is O and, by the previous argument, $Z\boldsymbol{A}(t)Z$ is O. Hence $\boldsymbol{A}(t)$ is SO.

As a result of this analysis, we conclude that if $\boldsymbol{A}(0)$ is a pentadiagonal SO matrix of the form (4.3), then $\boldsymbol{A}(t)$ will also be a pentadiagonal SO matrix. We need to verify two further results:

(i) that if the terms in the outermost diagonal, i.e. c_i, $i = 1, \dots, n-2$, are initially positive, then they stay positive; and

(ii) the inverse $\boldsymbol{B}(t) = (\boldsymbol{A}(t))^{-1}$ is a strictly positive matrix.

To prove (i), we note that when $\boldsymbol{A}(t)$ satisfies (6.1), the terms c_i satisfy

$$\dot{c}_i = (a_{i+2} - a_i)c_i, \quad i = 1, 2, \dots, n-2,$$

so that, as with the d_p, the c_i maintain their (strict) sign.

To prove (ii), we note that, since $\boldsymbol{A}(t)$ is SO, $\boldsymbol{B}(t) = (\boldsymbol{A}(t))^{-1}$ is O. Thus, to prove that $\boldsymbol{B}(t)$ is a strictly positive matrix, it is sufficient, following Markham (1970) as in § 4, to show that the corner term $b_{n1}(t)$ is positive. But when $\boldsymbol{A}(t)$ satisfies (6.1), then $\boldsymbol{B}(t)$ satisfies the same equation, i.e. $\dot{\boldsymbol{B}} = \boldsymbol{BS} - \boldsymbol{SB}$, the corner term b_{n1} satisfies

$$\dot{b}_{n1} = (a_{nn} - a_{11})b_{n1},$$

and hence maintains its sign.

We conclude that $\boldsymbol{A}(t)$ satisfies all the conditions for it to be factorized as in § 5, to give positive lengths, masses and stiffnesses.

7. Conclusions

Given a model of a vibrating cantilever beam with masses m_i, lengths l_i and spring stiffnesses k_i, it is possible to find a four-parameter family of isospectral beams. Given one member $\mathcal{S}$ of this family, other members may be found by applying a shifted $\boldsymbol{QR}$ factorization to the mass-reduced stiffness matrix $\boldsymbol{A}$ of $\mathcal{S}$, reversing $\boldsymbol{Q}$ and $\boldsymbol{R}$, and then factorizing the new matrix $\boldsymbol{A}'$. Alternatively, another family of beams may be constructed by setting up an isospectral flow equation for the mass-reduced stiffness matrix. We have chosen to consider just the case of a cantilever (clamped-free) beam, but there is no difficulty in adapting the analysis to other end conditions.

These two procedures, shifted QR factorization and Toda flow, provide families of beams that are isospectral to the original one, but the analysis leaves an important open question. As stated earlier, it is known that three spectra, corresponding to three different end conditions at one end of the beam, are necessary to specify the beam uniquely. These three spectra, which must satisfy a stringent set of inequalities stated in Gladwell (1984), may be chosen to be those corresponding to the right-hand end being free, pinned and sliding. This means that there is an infinite family of beams with the same two spectra corresponding to the right-hand end being free and being pinned. These two spectra are the spectrum of $\boldsymbol{A}$, $\sigma(\boldsymbol{A})$, and the spectrum, $\sigma_n(\boldsymbol{A})$, of the matrix obtained by deleting the last row and column of $\boldsymbol{A}$. The analysis given in this paper provides families of beams with just one spectrum, $\sigma(\boldsymbol{A})$, in common. We have so far been unable to find a way to construct a family of pentadiagonal SO matrices with two spectra, say, $\sigma(\boldsymbol{A})$ and $\sigma_n(\boldsymbol{A})$, in common.

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