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On isospectral spring–mass systems

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Abstract. The paper concerns an in-line system of masses $(m_i)_1^n$ connected to each other and to the end supports by ideal massless springs $(k_i)_1^{n+1}$. Four ways are given for constructing a system which is isospectral to a given one: by using the interchange $m_i \to k_{n-i+1}^{-1}$, $k_i \to m_{n-i+1}^{-1}$ for a cantilever ($k_{n+1} = 0$); by using the indeterminacy associated with the reduction to standard form; by using one or more shifted $\mathbf{LL}^{\mathrm{T}}$ factorizations and reversals; by using one or more shifted **QR** factorizations and reversals. It is shown that one may pass from any system to any isospectral system by a reduction to standard form, $n-1$ **QR** factorizations and reversals, and a reversed reduction to standard form.

1. Introduction

Two vibrating systems which have the same natural frequencies are said to be *isospectral*. In a recent paper, Gladwell and Morassi (1995) showed how to construct families of rods, in longitudinal or torsional vibration, which were isospectral. This paper takes up the same problem for discrete mass–spring systems, of the type shown in figure 1. Such systems may be considered to be actual systems made up of rigid masses and massless springs, or they may be thought of as finite-difference or finite-element approximations of continuous systems.

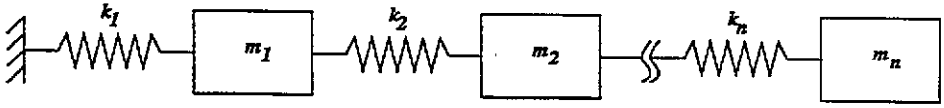

Figure 1. An in-line system of rigid masses and massless springs.

The natural frequencies $(\omega_r)_1^n$ and principal modes $(\mathbf{u}^{(i)})_1^n$ of the system shown in figure 1 are the solutions of the eigenvalue problem

$$(\mathbf{C} - \lambda\mathbf{M})\mathbf{u} = \mathbf{0} \qquad \lambda = \omega^2 \tag{1}$$

where

$$\mathbf{C} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \dots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ & & & -k_n & k_n + k_{n+1} \end{bmatrix}$$

$$\mathbf{M} = \mathrm{diag}(m_1, m_2, \dots, m_n). \tag{2}$$

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We will sometimes write

$$\mathbf{C} = \mathbf{C}(k_1, k_2, \ldots, k_{n+1}).$$

We will assume that the chain of masses and springs is unbroken so that

$$k_2, k_3, \ldots, k_n \qquad m_1, m_2, \ldots, m_n$$

are all strictly positive. We will often have to consider three cases:

(**S**) supported; $k_1 > 0, k_{n+1} > 0$

(**C**) cantilever; $k_1 > 0, k_{n+1} = 0$

(**F**) free; $k_1 = 0, k_{n+1} = 0$.

If two systems specified by $\mathbf{C}_1$, $\mathbf{M}_1$ and $\mathbf{C}_2$, $\mathbf{M}_2$ possess identical eigenvalues, i.e. are isospectral, we shall write

$$s(\mathbf{C}_1, \mathbf{M}_1) = s(\mathbf{C}_2, \mathbf{M}_2). \tag{3}$$

There are two almost trivial ways of obtaining isospectral pairs of systems as follows. First, since equation (1) is homogeneous of degree 1 in $\mathbf{C}$, $\mathbf{M}$, we have

$$s(c\mathbf{C}, c\mathbf{M}) = s(\mathbf{C}, \mathbf{M}) \tag{4}$$

for any positive constant c. We shall therefore normalize the problem by making $\sum_{i=1}^{n} m_i = 1$, or perhaps by making $m_1 = 1$. Secondly, if we physically turn the system around and renumber the masses and springs from the left, then we will not change the eigenvalues. Renumbering is equivalent to premultiplying and postmultiplying by

$$\mathbf{S} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

Thus equation (3) will hold if

$$\mathbf{C}_2 = \mathbf{S}\mathbf{C}_1\mathbf{S} \qquad \mathbf{M}_2 = \mathbf{S}\mathbf{M}_1\mathbf{S}.$$

To obtain non-trivial pairs of isospectral systems, we reduce equation (1) to standard form. We write

$$\mathbf{M} = \mathbf{G}^2 \qquad \mathbf{G}\mathbf{u} = \mathbf{x} \qquad \mathbf{A} = \mathbf{G}^{-1}\mathbf{C}\mathbf{G}^{-1} \tag{5}$$

so that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0. \tag{6}$$

We note that $\mathbf{A}$, like $\mathbf{C}$, is a Jacobian matrix, i.e. a symmetric tri-diagonal matrix with negative off-diagonal elements. First consider a cantilever system, i.e. one with the right hand free, so that $k_{n+1} = 0$. Now the matrix $\mathbf{C}$ may be factorized as

$$\mathbf{C} = \mathbf{E}\mathbf{K}\mathbf{E}^{\mathrm{T}}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \\ & & & 1 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{bmatrix}.$$

Write $\mathbf{K} = \mathbf{F}^2$, then

$$\mathbf{A} = \mathbf{G}^{-1}\mathbf{C}\mathbf{G}^{-1} = \mathbf{G}^{-1}\mathbf{E}\mathbf{F}^2\mathbf{E}^{\mathrm{T}}\mathbf{G}^{-1} = (\mathbf{G}^{-1}\mathbf{E}\mathbf{F})(\mathbf{F}\mathbf{E}^{\mathrm{T}}\mathbf{G}^{-1}) \tag{7}$$

so that $\mathbf{A}$ is the product of two matrices $\mathbf{G}^{-1}\mathbf{E}\mathbf{F}$ and its transpose $\mathbf{F}\mathbf{E}^{\mathrm{T}}\mathbf{G}^{-1}$. We need a simple result which is fundamental to this paper, and which is proved in the appendix.

Lemma 1. The matrices $\mathbf{A} = \mathbf{H}\mathbf{H}^{\mathrm{T}}$ and $\mathbf{A}^* = \mathbf{H}^{\mathrm{T}}\mathbf{H}$ have the same eigenvalues, except perhaps for zero.

Let us apply this lemma to our problem. We start with the cantilever system

$$(\mathbf{C} - \lambda\mathbf{M})\mathbf{u} = \mathbf{0}$$

where $\mathbf{C} = \mathbf{E}\mathbf{K}\mathbf{E}^{\mathrm{T}}$. We form the standard system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \qquad \mathbf{G}\mathbf{u} = \mathbf{x}.$$

We factorize $\mathbf{A}$ as in (7) and reverse the factors to form

$$(\mathbf{A}^* - \lambda\mathbf{I})\mathbf{y} = \mathbf{0}. \tag{8}$$

The new eigenvector $\mathbf{y}$ is related to $\mathbf{x}$ by

$$\mathbf{y} = \mathbf{H}^{\mathrm{T}}\mathbf{x} = \mathbf{F}\mathbf{E}^{\mathrm{T}}\mathbf{G}^{-1}\mathbf{x} = \mathbf{F}\mathbf{E}^{\mathrm{T}}\mathbf{u}. \tag{9}$$

Now we form a spring mass system which has the standard form (8); to do so, we must reverse the reduction to standard form. We note that

$$\mathbf{G}^{-2} = \mathbf{M}^{-1} \qquad \mathbf{F}^{-2} = \mathbf{K}^{-1}$$
$$\mathbf{A}^* = \mathbf{F}\mathbf{E}^{\mathrm{T}}\mathbf{G}^{-2}\mathbf{E}\mathbf{F} = \mathbf{F}\mathbf{E}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{E}\mathbf{F}$$

so that, putting $\mathbf{v} = \mathbf{F}\mathbf{y} = \mathbf{F}^2\mathbf{E}^{\mathrm{T}}\mathbf{u} = \mathbf{K}\mathbf{E}^{\mathrm{T}}\mathbf{u}$, we find

$$(\mathbf{E}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{E} - \lambda\mathbf{K}^{-1})\mathbf{v} = \mathbf{0}. \tag{10}$$

This is the eigenvalue equation for a reversed cantilever. We may verify this by noting that

$$\mathbf{S}\mathbf{E}\mathbf{S} = \mathbf{E}^{\mathrm{T}} \qquad \mathbf{S}^2 = \mathbf{I}$$

thus

$$\mathbf{S}\mathbf{E}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{E}\mathbf{v} = \mathbf{S}\mathbf{E}^{\mathrm{T}}\mathbf{S}\cdot\mathbf{S}\mathbf{M}^{-1}\mathbf{S}\cdot\mathbf{S}\mathbf{E}\mathbf{S}\cdot\mathbf{S}\mathbf{v} = \mathbf{E}\mathbf{K}_2\mathbf{E}^{\mathrm{T}}\cdot\mathbf{S}\mathbf{v}$$

so that we write equation (10) as

$$(\mathbf{C}_2 - \lambda\mathbf{M}_2)\mathbf{S}\mathbf{v} = \mathbf{0} \tag{11}$$

where

$$\mathbf{C}_2 = \mathbf{E}\mathbf{K}_2\mathbf{E}^{\mathrm{T}} \qquad \mathbf{K}_2 = \mathbf{S}\mathbf{M}^{-1}\mathbf{S} \qquad \mathbf{M}_2 = \mathbf{S}\mathbf{K}^{-1}\mathbf{S}. \tag{12}$$

This system relates to a cantilever with

$$k_i = m_{n-1+1}^{-1} \qquad m_i = k_{n-i+1}^{-1} \qquad i = 1, 2, \ldots, n$$

and

$$s(\mathbf{C}_2, \mathbf{M}_2) = s(\mathbf{C}, \mathbf{M}).$$

This pair was recently pointed out by Ram and Elhay (1994).

Now suppose that $k_1 = 0$ so that the spring–mass system in figure 1 is free; it will have a zero eigenvalue corresponding to the rigid body mode

$$\mathbf{u}^{(1)\mathrm{T}} = [1, 1, \dots, 1]. \tag{13}$$

We examine $\mathbf{A}^*$; it is $\mathbf{A}^* = \mathbf{F}\mathbf{E}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{E}\mathbf{F}$. Since $\mathbf{F}$ has its first row and column zero, so will $\mathbf{A}^*$. Thus $\mathbf{A}^*$ has the form

$$\mathbf{A}^* = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{*\prime} \end{bmatrix}.$$

The lemma states that $\mathbf{A}^*$ has the same eigenvalues as $\mathbf{A}$, apart perhaps from zero. Thus $\mathbf{A}^{*\prime}$ must have all the $n-1$ non-zero eigenvalues of $\mathbf{A}$, i.e. all the eigenvalues of the original system apart from the zero eigenvalue corresponding to the rigid-body mode. We note that $\mathbf{A}^*$ does have a zero eigenvalue, so that $\mathbf{A}$ and $\mathbf{A}^*$ have, in fact, all the same eigenvalues; but the eigenvector $\mathbf{y}$ of $\mathbf{A}^*$ given by (9) corresponding to the rigid-body mode (13) is identically zero because $\mathbf{E}^{\mathrm{T}}\mathbf{u}^{(1)}$ has zero in all but the first place, and $k_1 = 0$. We now construct a supported system which has the standard matrix $\mathbf{A}^{*\prime}$. By expanding the matrix product for $\mathbf{A}^*$ we see that $\mathbf{A}^{*\prime}$ may be written

$$\mathbf{A}^{*\prime} = \mathbf{F}'\mathbf{C}'\mathbf{F}'$$

where

$$\mathbf{C}' = \mathbf{C}(m_2^{-1}, m_3^{-1}, \dots, m_n^{-1}) \qquad \mathbf{F}' = \mathrm{diag}(f_2, f_3, \dots, f_n).$$

Thus on reversing the reduction to standard form we find

$$(\mathbf{C}' - \lambda\mathbf{M}')\mathbf{v}' = \mathbf{0} \qquad \mathbf{v}' = \mathbf{K}'\mathbf{E}'^{\mathrm{T}}\mathbf{u}$$

where $\mathbf{K}' = \mathrm{diag}(k_2, \dots, k_n)$, $\mathbf{M}' = (\mathbf{K}')^{-1}$, $\mathbf{v}' = \mathbf{K}'\mathbf{E}'^{\mathrm{T}}\mathbf{u}$ and $\mathbf{E}'^{\mathrm{T}}$ is obtained from $\mathbf{E}^{\mathrm{T}}$ by deleting the first row. Thus we can state

$$s(\mathbf{C}', \mathbf{M}') = s'(\mathbf{C}, \mathbf{M})$$

where s' means that the rigid-body mode is deleted.

In this section we have obtained a pair of isospectral cantilever systems, and a supported system which is isospectral to a free system apart from the rigid body mode. In the next section we will obtain further isospectral systems by considering the indeterminacy of the reduction to standard form.

2. The indeterminacy of the reduction to standard form

Given a system specified by matrices $\mathbf{C}$, $\mathbf{M} = \mathbf{G}^2$ there is a unique matrix

$$\mathbf{A} = \mathbf{G}^{-1}\mathbf{C}\mathbf{G}^{-1}$$

but, from a given positive semi-definite Jacobian matrix $\mathbf{A}$, we may construct an infinity of isospectral spring–mass systems, as we will now show.

The stiffness matrix $\mathbf{C}$ of (2) has the characterizing property

$$\mathbf{C}\{1, 1, \ldots, 1\} = \{k_1, 0, \ldots, 0, k_{n+1}\}.$$

We rewrite this equation in terms of $\mathbf{A}$:

$$\mathbf{A}\{g_1, g_2, \ldots, g_n\} = \mathbf{G}^{-1}\{k_1, 0, \ldots, 0, k_{n+1}\}$$
$$= \{g_1^{-1}k_1, 0, \ldots, 0, g_n^{-1}k_{n+1}\}.$$

Thus in order to find a spring–mass system we must take $\mathbf{A}$ and to find a *positive* solution $\mathbf{g} = \{g_1, g_2, \ldots, g_n\}$ to the equation

$$\mathbf{A}\mathbf{g} = \{\alpha, 0, 0, \ldots, 0, \beta\} \tag{14}$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta > 0$. Having found $\mathbf{g}$ we construct $\mathbf{G} = \mathrm{diag}(g_1, g_2, \ldots, g_n)$ and then the system is given by

$$\mathbf{C} = \mathbf{GAG} \qquad \mathbf{M} = \mathbf{G}^2. \tag{15}$$

To show that we can construct such a positive solution we establish:

Lemma 2. If $\mathbf{A}$ is a positive definite Jacobian matrix, then $\mathbf{A}^{-1} > \mathbf{0}$.

Note that we use $\mathbf{A}^{-1} > \mathbf{0}$ to mean that *each* element of the matrix $\mathbf{A}^{-1}$ is positive. Similarly if each element of a vector $\mathbf{b}$ is positive we say $\mathbf{b} > \mathbf{0}$; if each element is positive or zero and at least one is positive, then we write $\mathbf{b} \geqslant \mathbf{0}$. The proof is given in the appendix.

To cover the case when $\mathbf{A}$ is positive semi-definite we use:

Lemma 3. If $\mathbf{A}$ is a positive semi-definite Jacobian matrix, then we may find $\mathbf{x}$, unique except for a positive factor such that

$$\mathbf{A}\mathbf{x} = \mathbf{0} \qquad \mathbf{x} > \mathbf{0}.$$

The proof is also given in the appendix.

We now return to the construction of the isospectral systems. If $\mathbf{A}$ is non-singular we may choose α, β in equation (14) to be arbitrary positive quantities. This is equivalent to choosing arbitrary spring stiffness k_1, k_{n+1}; for when we solve equation (14) we find

$$k_1 = g_1\alpha \qquad k_{n+1} = g_n\beta.$$

Note that we may take one, but not both, of α, β to be zero; we have a two-parameter family of isospectral systems. If we demand that the reconstructed system to be a cantilever, so that $\beta = 0 = k_{n+1}$, then the solution is essentially unique; we can make it unique by taking $m_1 = 1$ or $\sum_{i=1}^{n} m_i = 1$.

If $\mathbf{A}$ is singular, we solve

$$\mathbf{A}\mathbf{g} = \mathbf{0}$$

to find $\mathbf{g} > \mathbf{0}$ and then construct $\mathbf{C}$, $\mathbf{M}$ from (15); again the solution is essentially unique.

In the next section we discuss how we may construct Jacobian matrices isospectral to a given one.

3. A matrix version of the Darboux lemma

Let $\mathbf{A}$ be the Jacobian matrix

$$\mathbf{A} = \begin{bmatrix} a_1 & -b_1 & & \\ -b_1 & a_2 & -b_2 & \\ \cdots & \cdots & \cdots & \cdots \\ & & -b_{n-1} & a_n \end{bmatrix} \tag{16}$$

with $b_i > 0$ and eigenvalues $(\lambda_i)_1^n$ satisfying $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. If $\mu < \lambda_1$, then the matrix $\mathbf{A} - \mu\mathbf{I}$ will be positive definite, and so may be factorized in the form

$$\mathbf{A} - \mu\mathbf{I} = \mathbf{L}\mathbf{L}^{\mathrm{T}}. \tag{17}$$

The matrix $\mathbf{L}$ will be a lower-triangular bi-diagonal matrix with negative off-diagonal terms, of the form

$$\mathbf{L} = \begin{bmatrix} l_{11} & & & & \\ l_{12} & l_{22} & & & \\ & l_{23} & l_{33} & & \\ & & \ddots & \ddots & \\ & & & l_{n-1,n} & l_{nn} \end{bmatrix}. \tag{18}$$

Let $P_0, P_1, \ldots, P_n$ be the Sturm sequence for $\mathbf{A} - \mu\mathbf{I}$ defined by

$$P_0 = 1 \qquad P_1(\mu) = a_1 - \mu \qquad P_{j+1}(\mu) = (a_{j+1} - \mu)P_j(\mu) - b_j^2 P_{j-1}(\mu) \tag{19}$$

for $j = 1, 2, \ldots, n-1$. A straightforward calculation shows that

$$\begin{aligned} l_{jj} &= \left(\frac{P_j}{P_{j-1}}\right)^{1/2} \qquad j = 1, \ldots, n \\ l_{j,j+1} &= -b_j \left(\frac{P_{j-1}}{P_j}\right)^{1/2} \qquad j = 1, \ldots, n-1. \end{aligned} \tag{20}$$

Now reverse the factors and form the Jacobian matrix $\mathbf{A}^*$ given by

$$\mathbf{A}^* - \mu\mathbf{I} = \mathbf{L}^{\mathrm{T}}\mathbf{L}. \tag{21}$$

By lemma 1, the matrix $\mathbf{L}^{\mathrm{T}}\mathbf{L} \equiv \mathbf{A}^* - \mu\mathbf{I}$ has the same eigenvalues as $\mathbf{L}\mathbf{L}^{\mathrm{T}} \equiv \mathbf{A} - \mu\mathbf{I}$; therefore $\mathbf{A}^*$ has the same eigenvalues as $\mathbf{A}$. If the eigenvector of $\mathbf{A}$ corresponding to λ is $\mathbf{x}$, the eigenvector of $\mathbf{A}^*$ corresponding to λ is $\mathbf{x}^* = \mathbf{L}^{\mathrm{T}}\mathbf{x}$, for

$$\begin{aligned} (\mathbf{A}^* - \lambda\mathbf{I})\mathbf{x}^* &= (\mathbf{L}^{\mathrm{T}}\mathbf{L} - (\lambda - \mu)\mathbf{I})\mathbf{L}^{\mathrm{T}}\mathbf{x} = \mathbf{L}^{\mathrm{T}}(\mathbf{L}\mathbf{L}^{\mathrm{T}} - (\lambda - \mu)\mathbf{I})\mathbf{x} \\ &= \mathbf{L}^{\mathrm{T}}(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0. \end{aligned}$$

Equation (20) shows that the factorization will not be real if μ lies in the range $\lambda_1 < \mu < \lambda_n$, for then some of the $P_j(\mu)$ will be negative; the factorization will break down completely if one of the $P_j(\mu)$, $j = 1, 2, \ldots, n-1$ is zero. (If $\mu > \lambda_n$, we may carry out the operations

$$\mathbf{A} - \mu\mathbf{I} = -\mathbf{L}\mathbf{L}^{\mathrm{T}} \qquad \mathbf{A}^* - \mu\mathbf{I} = -\mathbf{L}^{\mathrm{T}}\mathbf{L}.) \tag{22}$$

This situation is analogous to that found in Gladwell and Morassi (1995) for the continuous case; the Sturm–Liouville potential $\check{q}(x)$ found after one application of the Darboux lemma, involved (the second derivative of) a logarithm of a function $g(x)$ which could change sign, or have zeros, in its domain of definition—see equation (22) in that paper. In the continuous case we found that, if we wanted to construct an acceptable $q(x)$, we had to apply the Darboux lemma twice. We do the same here. Thus we factorize

$$\mathbf{A}^* - \mu\mathbf{I} \equiv \mathbf{L}^{\mathrm{T}}\mathbf{L} = \mathbf{L}^*\mathbf{L}^{*\mathrm{T}} \tag{23}$$

and then form $\mathbf{A}^{**}$ from

$$\mathbf{A}^{**} - \mu\mathbf{I} = \mathbf{L}^{*\mathrm{T}}\mathbf{L}^*. \tag{24}$$

But it is known that such a repeated use of $\mathbf{LL}^{\mathrm{T}}$ factorization and factor reversal is equivalent to one $\mathbf{QR}$ factorization and reversal. For the equation

$$\mathbf{L}^{\mathrm{T}}\mathbf{L} = \mathbf{L}^*\,\mathbf{L}^{*\mathrm{T}}$$

implies

$$\mathbf{Q} \equiv \mathbf{LL}^{*-\mathrm{T}} = \mathbf{L}^{-\mathrm{T}}\mathbf{L}^* = \mathbf{Q}^{-\mathrm{T}}$$

so that $\mathbf{QQ}^{\mathrm{T}} = \mathbf{I}$ and $\mathbf{Q}$ is orthogonal. Thus, denoting by $\mathbf{R}$ the upper triangular, tri-diagonal matrix

$$\mathbf{R} = \mathbf{L}^{*\mathrm{T}}\mathbf{L}^{\mathrm{T}} \tag{25}$$

we may write

$$\mathbf{A} - \mu\mathbf{I} = \mathbf{LL}^{\mathrm{T}} = (\mathbf{LL}^{*-\mathrm{T}})(\mathbf{L}^{*\mathrm{T}}\mathbf{L}^{\mathrm{T}}) = \mathbf{QR} \tag{26}$$

$$\mathbf{A}^{**} - \mu\mathbf{I} = \mathbf{L}^{*\mathrm{T}}\mathbf{L}^* = (\mathbf{L}^{*\mathrm{T}}\mathbf{L}^{\mathrm{T}})(\mathbf{L}^{-\mathrm{T}}\mathbf{L}^*) = \mathbf{RQ}. \tag{27}$$

The $\mathbf{QR}$ factorization does not suffer from the drawbacks of the $\mathbf{LL}^{\mathrm{T}}$ factorization which we noted above; it can be carried out for all μ, for example (Golub and Van Loan 1983, p 147) by multiplying $\mathbf{A} - \mu\mathbf{I}$ on the left by orthogonal Householder matrices which successively delete the parts of successive columns which lie below the diagonal, and so finally giving the factorization

$$\mathbf{Q}^{\mathrm{T}}(\mathbf{A} - \mu\mathbf{I}) = \mathbf{R}. \tag{28}$$

We note that $\mathbf{A}^{**}$ is a symmetric tri-diagonal matrix with non-positive off diagonal elements (b_{n-1}^{**} may be zero if μ is an eigenvalue of $\mathbf{A}$), which is isospectral to $\mathbf{A}$, and that if $\mathbf{Ax} = \lambda\mathbf{x}$, then

$$\mathbf{x}^{**} = \mathbf{Q}^{\mathrm{T}}\mathbf{x} \tag{29}$$

is a normalized eigenvector of $\mathbf{A}^{**}$, for

$$\begin{aligned}(\mathbf{A}^{**} - \lambda\mathbf{I})\mathbf{x}^{**} &= (\mathbf{RQ} - (\lambda - \mu)\mathbf{I})\mathbf{Q}^{\mathrm{T}}\mathbf{x}\\ &= (\mathbf{R} - (\lambda - \mu)\mathbf{Q}^{\mathrm{T}})\mathbf{x}\\ &= \mathbf{Q}^{\mathrm{T}}(\mathbf{QR} - (\lambda - \mu)\mathbf{I})\mathbf{x} = \mathbf{Q}^{\mathrm{T}}(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}\end{aligned} \tag{30}$$

$$\mathbf{x}^{**\mathrm{T}}\mathbf{x}^{**} = \mathbf{x}^{\mathrm{T}}\mathbf{QQ}^{\mathrm{T}}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{x} = 1.$$

As a matter of interest we note that the elements of $\mathbf{R}$, and of $\mathbf{A}^{**} - \mu\mathbf{I}$, written as in (16), but with double starred quantities, may be expressed in terms of the polynomials P_j and a new set of polynomials Q_j taking only positive values, given by

$$Q_0 = 1 \qquad Q_j(\mu) = P_j^2(\mu) + b_j^2 Q_{j-1}(\mu) \qquad j = 1, 2, \ldots, n. \tag{31}$$

Thus

$$r_{jj} = \left(\frac{Q_j}{Q_{j-1}}\right)^{1/2} \qquad j = 1, 2, \ldots, n-1 \qquad r_{nn} = \frac{P_n}{(Q_{n-1})^{1/2}} \tag{32}$$

$$r_{j,j+1} = -b_j \frac{(P_{j-1}Q_j + P_{j+1}Q_{j-1})}{P_j(Q_{j-1}Q_j)^{1/2}} \qquad j = 1, 2, \ldots, n-1 \tag{33}$$

$$r_{j,j+2} = b_j b_{j+1} \left(\frac{Q_{j-1}}{Q_j}\right)^{1/2} \qquad j = 1, 2, \ldots, n-2 \tag{34}$$

$$a_j^{**} - \mu = \frac{P_{j-1}Q_j^2 + b_j^2 P_{j+1} Q_{j-1}^2}{P_j Q_{j-1} Q_j} \qquad j = 1, \ldots, n-1 \qquad a_n^{**} - \mu = \frac{P_n P_{n-1}}{Q_{n-1}} \tag{35}$$

$$b_j^{**} = b_j \left(\frac{Q_{j-1}Q_{j+1}}{Q_j^2}\right)^{1/2} \qquad j = 1, 2, \ldots, n-2 \qquad b_{n-1}^{**} = b_{n-1}\left(\frac{P_n^2 Q_{n-2}}{Q_{n-1}^2}\right)^{1/2}. \tag{36}$$

We note that the expressions for $r_{j,j+1}$ and $a_j^{**} - \mu$ are invalid when $P_j(\mu) = 0$, but that the recurrence relations (19), (31) imply

$$\begin{aligned} P_{j-1}Q_j + P_{j+1}Q_{j-1} &= P_{j-1}(P_j^2 + b_j^2 Q_{j-1}) + \{(a_{j+1} - \mu)P_j - b_j^2 P_{j-1}\}Q_{j-1} \\ &= P_j\{P_{j-1}P_j + (a_{j+1} - \mu)Q_{j-1}\} \\ P_{j-1}Q_j^2 + b_j^2 P_{j+1} Q_{j-1}^2 &= P_{j-1}(P_j^2 + b_j^2 Q_{j-1})^2 + b_j^2\{(a_{j+1} - \mu)P_j - b_j^2 P_{j-1}\}Q_{j-1}^2 \\ &= P_j\{P_{j-1}P_j(P_j^2 + 2b_j^2 Q_{j-1}) + b_j^2(a_{j+1} - \mu)Q_{j-1}^2\} \end{aligned}$$

so that the offending P_j may be cancelled from the respective numerators and denominators.

When μ is an eigenvalue of $\mathbf{A}$, say $\mu = \lambda_j$, then equation (35), (36) show that $a_n^{**} = \lambda_j$, $b_{n-1}^{**} = 0$ so that $\mathbf{A}^{**}$ has the form

$$\mathbf{A}^{**} = \begin{bmatrix} \mathbf{A}_{n-1} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \lambda_j \end{bmatrix}.$$

This is not an acceptable form for the $\mathbf{A}$-matrix for the system (5), unless $\lambda_j = 0$, and we have already considered this case in section 2. We shall henceforth assume that μ is not an eigenvalue of $\mathbf{A}$. In that case we may use equation (30) to replace equation (29) by

$$\mathbf{x}^{**} = \frac{\mathbf{Rx}}{\lambda - \mu}. \tag{37}$$

This has the advantage of expressing x_n^{**} as a multiple of x_n alone; if we choose the signs of x_n, x_n^{**} to be positive, then

$$x_n^{(i)**} = \frac{|r_{nn}(\mu)|}{|\lambda_i - \mu|} x_n^{(i)}. \tag{38}$$

4. The isospectral set

Let $\mathcal{M}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ denote the set of $n \times n$ Jacobian matrices with eigenvalues $(\lambda_i)_1^n$. It is well known (Parlett 1980, Gladwell 1986) that any matrix $\mathbf{A}$ in $\mathcal{M}$ may be constructed uniquely from the eigenvalues $(\lambda_i)_1^n$ and the last components $(x_n^{(i)})_1^n$ of its normalized eigenvectors. The eigenvalues must be simple, i.e. $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, and the $x_n^{(i)}$ must be non-zero and satisfy $\sum_{i=1}^n (x_n^{(i)})^2 = 1$. This means that each member of $\mathcal{M}$ may be associated with a point in the (strictly) positive orthant of the unit n-sphere. (In more precise terms, $\mathcal{M}$ is a smooth $(n-1)$-dimensional manifold diffeomorphic to the strictly positive orthant of the unit n-sphere.) Several authors have discussed the isospectral flow of the matrix $\mathbf{A}$ on $\mathcal{M}$. This is linked with the theory of the Toda lattice and Lax pairs; a clear exposition, with references, may be found in Nanda (1985).

We are concerned with finding mass spring systems specified by $\mathbf{C}$, $\mathbf{M}$ that are isospectral to one specified by an initial pair $\mathbf{C}_0$, $\mathbf{M}_0$. One way to proceed is as follows:

(i) From $\mathbf{C}_0$, $\mathbf{M}_0$ construct $\mathbf{A}_0$ as in (5).

(ii) Find the eigenvalues of $\mathbf{A}_0$; call them $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(iii) Choose positive $(x_n^{(i)})_1^n$ satisfying $\sum_{i=1}^n (x_n^{(i)})^2 = 1$.

(iv) Construct the Jacobian matrix $\mathbf{A}$ with eigenvalues $(\lambda_i)_1^n$, last components of eigenvectors $(x_n^{(i)})_1^n$ using, say, the Lanczos algorithm (Golub and Boley (1977) or Gladwell (1986)).

(v) Choose $\alpha, \beta \geqslant 0$ with at least one positive if $\lambda_1 > 0$, and with both zero of $\lambda_1 = 0$.

(vi) Solve equation (14).

(vii) Construct $\mathbf{C}$, $\mathbf{M}$ from equation (15).

The shifted $\mathbf{QR}$ factorization and reversal described in section 3 provides an alternative which can be used to replace steps (ii)–(iv), as we now describe. If μ is not an eigenvalue, i.e. $\mu \neq \lambda_j$, $j = 1, 2, \ldots, n$, then the transformation $\mathbf{A} \to \mathbf{A}^{**}$ defines a nonlinear operator $\mathcal{G}$ from $\mathcal{M}$ into $\mathcal{M}$ such that

$$\mathcal{G}_\mu \mathbf{A} = \mathbf{A}^{**}. \tag{39}$$

We note that this operator is commutative, in that

$$\mathcal{G}_\mu \mathcal{G}_\nu \mathbf{A} = \mathcal{G}_\nu \mathcal{G}_\mu \mathbf{A}. \tag{40}$$

This follows immediately from (38) which shows that the last coefficients of the normalized eigenvectors of both matrices in (40) will be proportional to

$$\frac{x_n^{(i)}}{|(\lambda_i - \mu)(\lambda_i - \nu)|}.$$

Since they are proportional and the sum of the squares of each set of coefficients is unity, the coefficient in the two sets must be equal, and (40) follows.

We now show that we may pass from any matrix $\mathbf{A}$ in $\mathcal{M}$ to any other matrix $\mathbf{B}$ in $\mathcal{M}$ in $(n-1)$ applications of $\mathcal{G}_\mu$, i.e. we can find $\mu_1, \mu_2, \ldots, \mu_{n-1}$ such that

$$\mathcal{G}_{\mu_1} \mathcal{G}_{\mu_2} \ldots \mathcal{G}_{\mu_{n-1}} \mathbf{A} = \mathbf{B}. \tag{41}$$

In order to show this it is sufficient to show that we can pass from one set of last components $x_n^{(i)}$ to any other set $y_n^{(i)}$ in $n-1$ $\mathcal{G}_\mu$ operations. But equation (38) shows that this is equivalent to choosing $\mu_1, \mu_2, \ldots, \mu_{n-1}$ such that

$$\prod_{j=1}^{n-1} \frac{1}{|\lambda_i - \mu|} x_n^{(i)} \propto y_n^{(i)}.$$

This is equivalent to choosing the polynomial

$$P(\lambda) = K \prod_{j=1}^{n-1} (\lambda - \mu_j) \qquad \lambda > 0 \tag{42}$$

such that

$$|P(\lambda_i)| = x_n^{(i)}/y_n^{(i)} \qquad i = 1, 2, \ldots, n. \tag{43}$$

If we choose the μ_j so that $\lambda_{j-1} < \mu_j < \lambda_j$, then

$$P(\lambda_i) = (-1)^{n-i} |P(\lambda_i)|$$

so that

$$P(\lambda_i) = (-1)^{n-i} x_n^{(i)}/y_n^{(i)}. \tag{44}$$

But there is a unique such polynomial $P(\lambda)$ of degree $n-1$ taking values of opposite signs at n points λ_i, and it will have $n-1$ roots μ_j satisfying $\lambda_{j-1} < \mu_j < \lambda_j$. In particular we note that if

$$\mathcal{G}_\mu \mathbf{A} = \mathbf{A}^{**} \tag{45}$$

then we can find $\mu_1, \mu_2, \ldots, \mu_{n-1}$ such that

$$\mathcal{G}_{\mu_1} \mathcal{G}_{\mu_2} \ldots \mathcal{G}_{\mu_{n-1}} \mathbf{A}^{**} = \mathbf{A} \tag{46}$$

and so find the inverse $\mathcal{G}_\mu^{-1} \mathbf{A}^{**}$. We note also that we can pass from $\mathbf{A}$ back to $\mathbf{A}$ in $n-1$ steps.

5. Conclusions

We have described four ways in which we can form mass–spring systems isospectral to a given one:

(i) by the interchange $m_i \to k_{n-i+1}^{-1}$, $k_i \to m_{n-i+1}^{-1}$ for a cantilever system;
(ii) by using the indeterminacy associated with the reduction to standard form;
(iii) by using one or more $\mathbf{LL}^T$ transformations and reversals given by (8), (21) when $\mu < \lambda_1$, and by (22) if $\mu > \lambda_n$ and;
(iv) by using one or more $\mathbf{QR}$ transformations and reversals.

We have shown that we can pass from any Jacobian matrix $\mathbf{A}$ to any other isospectral Jacobian matrix $\mathbf{B}$ by means of $n-1$ $\mathcal{G}_\mu$ operations. This implies that, starting from any system $\mathbf{C}_0, \mathbf{M}_0$, we can perform a reduction to standard form, $n-1$ $\mathcal{G}_\mu$ operations and a reversal of the reduction to standard form to reach any isospectral system $\mathbf{C}, \mathbf{M}$.

Appendix

Lemma 1. The matrices $\mathbf{A} = \mathbf{H}\mathbf{H}^{\mathrm{T}}$ and $\mathbf{A}^* = \mathbf{H}^{\mathrm{T}}\mathbf{H}$ have the same eigenvalues, except perhaps for zero.

Proof. Suppose $\lambda \neq 0$ is an eigenvalue of $\mathbf{A}$. Then $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$. Thus $\mathbf{H}\mathbf{H}^{\mathrm{T}}\mathbf{x} = \lambda\mathbf{x}$ so that $\mathbf{y} \equiv \mathbf{H}^{\mathrm{T}}\mathbf{x} \neq \mathbf{0}$. But $\mathbf{H}\mathbf{H}^{\mathrm{T}}\mathbf{x} = \lambda\mathbf{x}$ implies $\mathbf{H}^{\mathrm{T}}(\mathbf{H}\mathbf{H}^{\mathrm{T}}\mathbf{x}) = \mathbf{H}^{\mathrm{T}}\mathbf{H}(\mathbf{H}^{\mathrm{T}}\mathbf{x}) = \lambda\mathbf{H}^{\mathrm{T}}\mathbf{x}$, i.e. $\mathbf{A}^*\mathbf{y} = \lambda\mathbf{y}$. Since $\mathbf{y} \neq \mathbf{0}$, λ is an eigenvalue of $\mathbf{A}^*$. Thus non-zero eigenvalues of $\mathbf{A}$ are eigenvalues of $\mathbf{A}^*$ and, by reversing the argument, we may equally show that non-zero eigenvalues of $\mathbf{A}^*$ are eigenvalues of $\mathbf{A}$. □

Lemma 2. If $\mathbf{A}$ is a positive definite Jacobian matrix, then $\mathbf{A}^{-1} > \mathbf{0}$.

Proof. We use induction. The result is true for $n = 1$. Suppose it is true for $n - 1$. Thus we may write

$$\mathbf{A}_n = \begin{bmatrix} \mathbf{A}_{n-1} & -\mathbf{b} \\ -\mathbf{b}^{\mathrm{T}} & a_{nn} \end{bmatrix} \qquad \mathbf{A}_n^{-1} = \begin{bmatrix} \mathbf{H} & \mathbf{h} \\ \mathbf{h}^{\mathrm{T}} & h_{nn} \end{bmatrix} \tag{47}$$

where $\mathbf{b} \geqslant \mathbf{0}$. Since $\mathbf{A}_n^{-1}$ is positive definite, $h_{nn} > 0$. The result holds for $\mathbf{A}_{n-1}$, i.e. $\mathbf{A}_{n-1}^{-1} > \mathbf{0}$. Now

$$\begin{bmatrix} \mathbf{A}_{n-1} & -\mathbf{b} \\ -\mathbf{b}^{\mathrm{T}} & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{H} & \mathbf{h} \\ \mathbf{h}^{\mathrm{T}} & h_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{0} \\ -\mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}$$

so that

$$\mathbf{A}_{n-1}\mathbf{H} - \mathbf{b}\mathbf{h}^{\mathrm{T}} = \mathbf{I} \qquad \mathbf{A}_{n-1}\mathbf{h} - \mathbf{b}h_{nn} = \mathbf{0}$$

and thus

$$\mathbf{h} = \mathbf{A}_{n-1}^{-1}\mathbf{b}h_{nn} > \mathbf{0} \qquad \mathbf{H} = \mathbf{A}_{n-1}^{-1}\mathbf{b}\mathbf{h}^{\mathrm{T}} + \mathbf{A}_{n-1}^{-1} > \mathbf{0}$$

so that $\mathbf{A}_n^{-1} > \mathbf{0}$ as required. (Note that $\mathbf{A}_{n-1}^{-1} > \mathbf{0}$, $\mathbf{b} \geqslant \mathbf{0}$ implies $\mathbf{A}_{n-1}^{-1}\mathbf{b} > \mathbf{0}$.) □

Lemma 3. If $\mathbf{A}$ is a positive semi-definite Jacobian matrix, then we may find $\mathbf{x}$, unique except for a positive factor such that

$$\mathbf{A}\mathbf{x} = \mathbf{0} \qquad \mathbf{x} > \mathbf{0}.$$

Proof. First we note that the eigenvalues of a Jacobian matrix are simple, so that $\mathbf{A}$ can have only one zero eigenvalue and corresponding eigenvector. Write $\mathbf{A} \equiv \mathbf{A}_n$ as in (47). Since $\mathbf{A}_n$ is positive semi-definite, $\mathbf{A}_{n-1}$ must be positive definite. (Two successive members of the Sturm sequence of principal numbers of $\mathbf{A}_n$ evaluated for $\lambda = 0$ cannot be simultaneously zero.) Therefore, by lemma 2, $\mathbf{A}_{n-1}^{-1} > \mathbf{0}$.

$$\mathbf{A}_n\mathbf{x} = \begin{bmatrix} \mathbf{A}_{n-1} & -\mathbf{b} \\ -\mathbf{b}^{\mathrm{T}} & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{x}_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}$$

so that

$$\mathbf{A}_{n-1}\mathbf{x}_{n-1} - \mathbf{b}x_n = \mathbf{0} \qquad -\mathbf{b}^{\mathrm{T}}\mathbf{x}_{n-1} + a_{nn}x_n = 0.$$

Choose $x_n > 0$, then

$$\mathbf{x}_{n-1} = \mathbf{A}_{n-1}^{-1}\mathbf{b}x_n > 0$$

as required. □

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