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Inner totally positive matrices

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Abstract

Suppose $A \in M_n$ is a staircase matrix with row and column staircase sequences $\rho = \{\rho(1), \rho(2), \ldots, \rho(n)\}$, $\gamma = \{\gamma(1), \gamma(2), \ldots, \gamma(n)\}$. A minor $A(\alpha; \beta) = \det(A[\alpha|\beta])$ with $\alpha = \{\alpha_1, \ldots, \alpha_k\}, \beta = \{\beta_1, \ldots, \beta_k\}$ is said to be an inner minor of A if $\alpha_i \leqslant \gamma(\beta_i), \beta_i \leqslant \rho(\alpha_i)$ for $i = 1, 2, \ldots, k$. A is said to be *inner totally positive* (ITP) if every inner minor of A is positive. We prove that A is ITP if $A(\alpha; \beta) > 0$ for all inner minors with $\alpha, \beta \in Q^0_{k,n}, k = 1, 2, \ldots, n$. Also $A \in M_n$ is ITP iff it is totally non-negative and its *extreme* inner minors are positive. We show that an ITP matrix may be reduced by similarity transformations to an ITP band matrix, and may alternatively be filled-in by similarity transformations to become a TP matrix. In both the reduction and the filling in, the matrix is ITP at each stage. The analysis is applied to some inverse eigenvalue problems for band matrices.

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1. Notation and basic theory

Let M_n denote the set of real $n \times n$ matrices, and S_n denote the subset of symmetric matrices.

Following Ando [1], we let $Q_{k,n}$ denote the set of strictly increasing sequences of k integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ taken from $\{1, 2, \ldots, n\}$. We denote the submatrix of

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$A \in M_n$ lying in rows indexed by $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and columns indexed by $\beta = \{\beta_1, \ldots, \beta_k\}$, by $A[\alpha|\beta]$, and write $A[\alpha|\alpha] = A[\alpha]$. When α, β have the same length, we write

$$\det A[\alpha|\beta] = A(\alpha; \beta).$$

When $\alpha \in Q_{k,n}$, $\beta \in Q_{l,n}$ and $\alpha \cap \beta = 0$, then their union $\alpha \cup \beta$ is rearranged increasingly to become a member of $Q_{k+\ell,n}$.

If $\alpha \in Q_{k,n}$, its *dispersion* number $d(\alpha)$ is defined by

$$d(\alpha) = \sum_{i=1}^{k-1} \{\alpha_{i+1} - (\alpha_i + 1)\} = \alpha_k - \alpha_1 - (k-1),$$

with the convention that $d(\alpha) = 0$ for $\alpha \in Q_{1,n}$. Thus $d(\alpha) = 0$ means that α consists of k consecutive integers. We denote the subset of $Q_{k,n}$ consisting of those α with $d(\alpha) = 0$, by $Q^0_{k,n}$.

The elements of $Q_{k.n}$ are partially ordered as follows: if $\alpha, \beta \in Q_{k,n}$ and $\alpha_i \leqslant \beta_i$ for $1 \leqslant i \leqslant k$ then we say $\alpha \leqslant \beta$; if $\alpha \leqslant \beta$ and $\alpha_i < \beta_i$ for some i, $1 \leqslant i \leqslant k$, then we say $\alpha < \beta$. The infimum of $Q_{k,n}$ is denoted by $\theta^{(k)} = \{1, 2, \ldots, k\}$.

A matrix $A \in M_n$ is said to be *totally positive* (TP) (*totally non-negative* (TN)) if all the minors $A(\alpha; \beta)$ are positive (non-negative). It is said to be NTN if it is non-singular (invertible) and TN. It is said to be *oscillatory* (O) if it is TN *and* a power of A, A^m is TP. It is known that A is O iff it is NTN and $a_{i,i+1}, a_{i+1,i} > 0$ for $i = 1, 2, \ldots, n-1$.

We recall some results concerning total positivity and non-negativity. Following Cryer [2] in his Remark 6.1, we may state

Theorem A. *If $A \in M_n$ is non-singular, then it is NTN if $A(\alpha; \beta) \geqslant 0$ for all α, β such that $\alpha \in Q_{k,n}$ and $\beta \in Q^0_{k,n}$ (or $\alpha \in Q^0_{k,n}$ and $\beta \in Q_{k,n}$), for $k = 1, 2, \ldots, n$.*

The counterexample

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

shows that for A to be NTN it is insufficient to have $A(\alpha; \beta) \geqslant 0$ for all $\alpha, \beta \in Q^0_{k,n}$.

Following Theorem 2.5 of Ando [1] we have

Theorem B. *$A \in M_n$ is TP if $A(\alpha; \beta) > 0$ for all α, β such that $\alpha, \beta \in Q^0_{k,n}$, $k = 1, 2, \ldots, n$.*

Thus A is TP iff $A(\alpha; \beta) > 0$ for all consecutive k-tuples α and β.

The following theorem is a simple consequence of Theorem 8 (p. 93) of Gantmacher and Krein [3].

Theorem C. *If $A \in M_n$ is NTN, then all its principal minors are positive.*

Theorems B and C enable us to give an immediate proof of a well known result for which (to the author's knowledge) there have before been only long and complicated proofs. (See for example the 2-page proof in Gladwell [4].)

Theorem D. *If $A \in M_n$ is TN then it is TP if the corner minors $A(n-p+1,\ldots,n;$ $1,2,\ldots,p)$ and $A(1,2,\ldots,p;n-p+1,\ldots,n)$ are positive for $p=1,2,\ldots,n$.*

Proof. Suppose $A \in M_n$ is TN. If $\alpha,\beta \in Q^0_{k,n}$ then $\alpha=\{i,i+1,\ldots,i+k-1\}$, $\beta=\{j,j+1,\ldots,j+k-1\}$. Suppose $i \geqslant j$; then $A(\alpha;\beta)$ is a principal minor of the NTN matrix $A[i-j+1,\ldots,n|1,\ldots,n-i+j]$ and so is positive by Theorem C. The case $i<j$ may be treated similarly. Theorem B now asserts that A is TP. □

Cryer [5] proves

Theorem E. *If $A \in M_n$ is NTN then it has a LU factorization $A=LU$ where L is lower triangular with unit diagonal, U is upper triangular and both are NTN.*

Gladwell [4] proved

Theorem F. *If $A \in S_n$, P is one of the properties NTN, O, TP; μ is not an eigenvalue of A, $A-\mu I=QR$ where Q is orthogonal, R is upper triangular with positive diagonal, and $A'-\mu I=RQ$, then A' has the same property P.*

Gladwell [6] proved

Theorem G. *If $A(0) \in S_n$, P is one of the properties NTN, O, TP, and $A(t)$ flows under the Toda flow $\dot{A}=AS-SA=[A,S]$, where $S=A^{+T}-A^{+}$, and A^{+} is the upper triangle of A, then $A(t)$ has the same property P for all t.*

Note that Theorems F and G hold for $A \in S_n$, not for all $A \in M_n$.

Gasca et al. [7] introduced the class of *almost strictly totally positive* (ASTP) matrices. Gasca and Peña [8] gave further characterizations of such matrices. When we wrote the original version of this paper we were unaware that the class of inner totally positive (ITP) matrices that we had introduced was identical to that of the *non-singular almost strictly totally positive* (NASTP) matrices, i.e., $A \in M_n$ is ITP iff it is NASTP. In fact, a referee pointed out that our Theorem 2.1 is a consequence of Theorem 3.1 of [7], and Theorem 2.2 is a consequence of (1) and (3) of Theorem 3.3 of [8]. We have retained the proofs of Theorems 2.1 and 2.2 because we believe that the concepts of *inner minor* and *extreme* minor, implicit in [7,8], but explicitly defined in this paper, clarify the analysis and simplify the proofs. We have retained the term *inner totally positive* because it has the merit of indicating that it is the inner minors that are the ones that are positive.

2. Inner totally positive matrices

A sequence $\gamma = \{\gamma(1), \gamma(2), \ldots, \gamma(n)\}$ is a *staircase sequence* if it is nondecreasing and satisfies $\gamma(i) \geqslant i$, $i = 1, 2, \ldots, n$. Let ρ (for row) and γ (for column) be staircase sequences. A matrix $A \in M_n$ is called a ρ, γ *staircase matrix* if $a_{i,j} = 0$ when $i > \gamma(j)$ *or* $j > \rho(i)$. If A is NTN, it is a (ρ, γ) staircase matrix when one defines

$$\rho(i) = \max\{j : a_{ij} > 0\} \quad \text{and} \quad \gamma(j) = \max\{i : a_{i,j} > 0\}.$$

If $\alpha, \beta \in Q_{k,n}$, a minor $A(\alpha; \beta)$ is called an *inner minor* of A if $\alpha_i \leqslant \gamma(\beta_i)$, $\beta_i \leqslant \rho(\alpha_i)$, $i = 1, 2, \ldots, k$. If $A(\alpha; \beta)$ is *no*t an inner minor, i.e., if $\alpha_i > \gamma(\beta_i)$ or $\beta_i > \rho(\alpha_i)$ for some $i = 1, 2, \ldots, k$, it is said to be an *outer* minor. If $A(\alpha; \beta)$ is an outer minor, it is zero, if A is an inner minor, it may or may not be zero. If all the inner minors of A are (strictly) positive then A is said to be *inner totally positive*.

Since $\det A = A(\theta^{(n)}; \theta^{(n)})$ is always an inner minor of A, an ITP matrix is NTN. If $\gamma(i) = i$ for some $i = 1, 2, \ldots, n-1$, then $A[i+1, \ldots, n|1, 2, \ldots, i] = 0$, so that A is reducible, as it is if $\rho(i) = i$ for some $i = 1, 2, \ldots, n-1$. We shall therefore often assume that $\gamma(i) \geqslant i+1$, $\rho(i) \geqslant i+1$ for $i = 1, 2, \ldots, n-1$. If A is ITP with $\rho(i) \geqslant i+1$, $\gamma(i) \geqslant i+1$, $i = 1, 2, \ldots, n-1$ then it is O. On the otherhand

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

is O but not ITP. However, if A is O and *tridiagonal* then it is ITP; all the inner minors are either codiagonal entries $a_{i,i+1}$, $a_{i+1,i}$, which are positive; principal minors, which are positive; or products of such quantities.

We show in this section how Theorems B, D, E and F may be generalized to ITP matrices.

We start by mimicking the proof of Theorem 2.5 of Ando [1] (and so of Theorem B) to prove

Theorem 2.1. *Let $A \in M_n$ be a ρ, γ-staircase matrix. A is ITP if $A(\alpha; \beta) > 0$ for all inner minors such that $\alpha, \beta \in Q^0_{k,n}$, $k = 1, 2, \ldots, n$.*

Proof. Let us prove that for inner minors

$$A(\alpha; \beta) > 0 \quad \text{for } \alpha, \beta \in Q_{k,n},\ k = 1, 2, \ldots, n, \tag{2.1}$$

by induction on k. When $k = 1$, this is trivial, because $Q_{1,n} = Q^0_{1,n}$. Assume that (2.1) is true with $k - 1$ ($k \geqslant 2$) in place of k. First fix $\alpha \in Q_{k,n}$ with $d(\alpha) = 0$, i.e., $\alpha \in Q^0_{k,n}$, and let us prove (2.1) with this α by induction on $\ell = d(\beta)$. When $\ell = 0$ this follows by the assumption of the theorem. Suppose $A(\alpha; \delta) > 0$ for all inner minors whenever $\delta \in Q_{k,n}$ and $d(\delta) \leqslant \ell - 1$, with $\ell \geqslant 1$. Take $\beta \in Q_{k,n}$ with $d(\beta) = \ell$ and such that $A(\alpha; \beta)$ is an inner minor.

Let $\tau = \{\beta_2, \ldots, \beta_{k-1}\}$, and use the identity

$$A(\omega; \tau \cup \{p\})A(\alpha; \tau \cup \{\beta_1, \beta_k\}) = A(\omega; \tau \cup \{\beta_k\})A(\alpha; \tau \cup \{\beta_1, p\}) + A(\omega; \tau \cup \{\beta_1\})A(\alpha; \tau \cup \{p, \beta_k\}) \quad (2.2)$$

(from Section 2 of Ando [1]) for any $\omega \in Q_{k-1,n}$ with $\omega \subset \alpha$. Each term in this equation is the product of a minor of order $k - 1$ and a minor of order k. We need to choose p so that $A(\omega; \tau \cup \{p\})$ is an inner minor; since it is of order $k - 1$ it will be positive by hypothesis. We then need one of the minors of order $k - 1$ on the right to be an inner minor (and so positive) and its multiplier, a minor of order k, to be an inner minor with the dispersion number for the columns satisfying $d \leqslant \ell - 1$; it too will then be positive by hypothesis.

There are two cases:

(i) $\beta_{m+1} \leqslant \rho(\alpha_m)$ for all $m = 1, 2, \ldots, k - 1$. Since $d(\beta) = \ell \geqslant 1$ we can choose p such that $\beta_i < p < \beta_{i+1}$ for some i, $1 \leqslant i \leqslant k - 1$, and let $\tau = \{\beta_2, \ldots, \beta_{k-1}\}$ then $d(\tau \cup \{p, \beta_k\}) \leqslant \ell - 1$ and $d(\tau \cup \{\beta_1, p\}) \leqslant \ell - 1$. Consider the two sequences in $A(\alpha; \tau \cup \{p, \beta_k\})$:

$$\begin{matrix} \alpha_1 & \alpha_2 & \ldots & \alpha_{i-1} & \alpha_i & \alpha_{i+1} & \ldots & \alpha_k \\ \beta_2 & \beta_3 & & \beta_i & p & \beta_{i+1} & \ldots & \beta_k \end{matrix}$$

They do correspond to an inner minor:

$$\beta_{m+1} \leqslant \rho(\alpha_m),\ m = 1, 2, \ldots, i - 1,\ \text{by hypothesis;}$$

$\alpha_m \leqslant \gamma(\beta_m) \leqslant \gamma(\beta_{m+1})$, $m = 1, 2, \ldots, i - 1$ because γ is a nondecreasing sequence. Also $p < \beta_{i+1} \leqslant \rho(\alpha_i)$, and $\alpha_i \leqslant \gamma(\beta_i) \leqslant \gamma(p)$. Now consider the minors of order $k - 1$; all are inner minors and so positive. The remaining minor $A(\alpha; \tau \cup (\beta_1, p\})$ is either inner or not; if it is, it has $d(\tau \cup \{\beta_1, p\}) \leqslant \ell - 1$, and so is positive by hypothesis; if it is not, it is zero. We conclude from (2.2) that $A(\alpha; \beta) \equiv A(\alpha; \tau \cup \{\beta_1, \beta_k\}) > 0$.

(ii) There is a smallest integer m such that $1 \leqslant m < k - 1$ and $\beta_{m+1} > \rho(\alpha_m)$. In that case $\beta_{m+1} > \rho(\alpha_i)$ for $i = 1, 2, \ldots, m$, so that $A[\alpha_1, \ldots, \alpha_m | \beta_{m+1}, \ldots, \beta_k] = 0$ and

$$A(\alpha; \beta) = A(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_m)A(\alpha_{m+1}, \ldots, \alpha_k; \beta_{m+1}, \ldots, \beta_k).$$

Each of the minors on the right is an inner minor of order $k - 1$ or less, and so is positive.

This proves (2.1) when $\alpha \in Q_{k,n}$ and $d(\alpha) = 0$. Apply the same argument row-wise to conclude that (2.1) is true for all $\alpha, \beta \in Q_{k,n}$. $\square$

The matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

is NTN and its inner *corner minors* (see Theorem D for the definition of a corner minor) are positive (the only inner corner minors in the lower triangle are $A(2, 3, 4, 5; 1, 2, 3, 4) = 1$, and $\det(A) = 1$) but it is not ITP, because the inner minor $A(4, 5; 2, 3) = 0$. This means that Theorem D must be modified: we must find a generalization for the concept of a corner minor.

A minor $A(\alpha; \beta)$ with $\alpha, \beta \in Q_{p,n}$ and $\alpha \geqslant \beta$ is called an *extreme* minor of A if it is an inner minor, and there is no inner minor $A(\alpha'; \beta)$ with $\alpha' > \alpha$, and no inner minor $A(\alpha; \beta')$ with $\beta' < \beta$; in short α cannot be increased nor β decreased and the minor still be an inner minor. If $\alpha \leqslant \beta$ then $A(\alpha; \beta)$ is an extreme minor if it is an inner minor, and there is no inner minor $A(\alpha'; \beta)$ with $\alpha' < \alpha$, and no inner minor $A(\alpha; \beta')$ with $\beta' > \beta$; α cannot be decreased nor β increased and the minor still be an inner minor.

Suppose $\alpha, \beta \in Q_{p,n}$, $\alpha \geqslant \beta$ and $A(\alpha; \beta)$ is an extreme minor. Suppose $d(\beta) \neq 0$. Then there exist $\beta_i, \beta_{i+1} \in \beta$ such that $\beta_{i+1} > \beta_i + 1$. We must have $\gamma(\beta_i) < \gamma(\beta_{i+1})$, for if $\gamma(\beta_i) = \gamma(\beta_{i+1})$ then $\gamma(\beta_i + 1) = \gamma(\beta_{i+1})$ so that we may replace β_{i+1} in β by $\beta_i + 1$, contrary to the hypothesis that $A(\alpha; \beta)$ is an extreme minor. Moreover, $\alpha_{i+1} > \gamma(\beta_i + 1)$ since, if not, again we could replace β_{i+1} by $\beta_i + 1$. Thus $\alpha_{i+1} > \gamma(\beta_i)$ so that

$$A[\alpha_{i+1}, \ldots, \alpha_p | \beta_1, \ldots, \beta_i] = 0$$

and

$$A(\alpha; \beta) = A(\alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_i) A(\alpha_{i+1}, \ldots, \alpha_p; \beta_{i+1}, \ldots, \beta_p)$$

and both minors are extreme since $A(\alpha; \beta)$ is extreme. We may thus express any extreme minor $A(\alpha; \beta)$ with $\beta \in Q_{p,n}$ as a product of extreme minors $A(\alpha; \beta)$ with $\beta \in Q^0_{p,n}$. We may apply the same argument to rows, and thus express any extreme minor $A(\alpha; \beta)$ with $\alpha, \beta \in Q_{p,n}$ as a product of extreme minors $A(\alpha; \beta)$ with each $\alpha, \beta \in Q^0_{p,n}$. This means that henceforth we may consider only extreme minors $A(\alpha; \beta)$ with consecutive p-tuples.

We may now state and prove the generalization of Theorem D.

Theorem 2.2. *If $A \in M_n$ is TN then it is ITP if all the extreme minors $A(\alpha; \beta)$, $\alpha, \beta \in Q^0_{p,n}\, p = 1, 2, \ldots, n$ are positive.*

Proof. Recall that $\det A = A(\theta^{(n)}; \theta^{(n)})$ is an extreme minor so that A is necessarily NTN. Suppose $A(\alpha; \beta)$, $\alpha, \beta \in Q^0_{p,n}$ is an inner minor that is not extreme. Write

$$\alpha = \{i, i+1, \ldots, i+p-1\}, \quad \beta = \{j, j+1, \ldots, j+p-1\}$$

and suppose $\alpha \geqslant \beta$, i.e., $i \geqslant j$.

Consider the matrix

$$B = A[i - j + 1, \dots, n|1, 2, \dots, n - i + j].$$

Either $\det B$ is an inner minor of A, in which case it is extreme, so that $A(\alpha; \beta)$ is a principal minor of the NTN matrix B. *Or* $\det B$ is not an inner minor of A. This means that

$$i - j + k > \gamma(k)$$

for one or more $k = 1, 2, \dots, n - i + j$.

Since $i - j + k \leqslant \gamma(k)$ for $k = j, j + 1, \dots, j + p - 1$, there is a maximal interval $[M, N]$ around $\{j, j + 1, \dots, j + p - 1\}$ such that $i - j + k \leqslant \gamma(k)$ for $M \leqslant k \leqslant N$; M cannot be decreased nor N increased. Either $M = 1$, or $M > 1$ and $i - j + M - 1 > \gamma(M - 1)$; either $N = n - i + j$, or $N < n - i + j$ and $i - j + N + 1 > \gamma(N + 1)$. Thus $A(i - j + M, \dots, i - j + N; M, \dots, N)$ is an extreme minor and $A(\alpha; \beta)$ is a principal minor of the NTN matrix $A[i - j + M, \dots, i - j + N|M, \dots, N]$. $\square$

Note that if A is a full lower diagonal matrix then $\gamma(i) = n, \rho(i) = 1, i = 1, 2, \dots, n$, and the extreme minors are the corner minors $A(n - p + 1, \dots, n; 1, 2, \dots, p)$, $p = 1, 2, \dots, n$; if these minors are positive then A is $\triangle$TP (strictly). Note that Theorem 3.1 of Cryer [5] does not assume that A is TN, and requires $A(\alpha; \theta^{(p)}) > 0$ for all $\alpha \in Q^0_{p,n}$, $p = 1, 2, \dots, n$.

The generalization of Theorem E is:

Theorem 2.3. *If $A \in M_n$ is ITP then it has a factorization $A = LU$; L, U are ITP and so is $B = UL$.*

Proof. This follows from Corollary 4.2 of [8]. $\square$

We may now use this to show that if A, B are ITP then so is AB.

It is known that if A is TP and B is NTN, then AB is TP. By contrast, it is *not* true to state that if A is ITP and B is NTN, then AB is ITP, as is demonstrated by the counterexample

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

AB is not ITP, because $AB(2, 3; 1, 2) = 0$.

We may extend Theorem F to ITP matrices:

Theorem 2.4. *If $A \in S_n$ is ITP, μ is not an eigenvalue of A, $A - \mu I = QR$, where Q is orthogonal and R is upper triangular with positive diagonal, $A' - \mu I = RQ$, then A' is ITP.*

Proof. Since A is NTN, Theorem F states that A' is NTN. A and A' have the same stair case sequence γ ($\rho = \gamma$ because $A \in S_n$). Since

$$RA = R(\mu I + QR) = (\mu I + RQ)R = A'R$$

we have

$$\sum R(\alpha; \delta)A(\delta; \beta) = \sum A'(\alpha; \delta)R(\delta; \beta). \tag{2.3}$$

Suppose $A'(\alpha; \beta)$ is an extreme minor of A' with $\alpha \geqslant \beta$, then $A(\alpha; \beta)$ is an extreme minor of A. The summation on the left of (2.3) is over all $\delta \geqslant \alpha$, that on the right over all $\delta \leqslant \beta$. But $A(\delta; \beta) = 0$ if $\delta > \alpha$, and $A'(\alpha; \delta) = 0$ if $\delta < \beta$, so that

$$R(\alpha; \alpha)A(\alpha; \beta) = A'(\alpha; \beta)R(\beta; \beta)$$

and $R(\alpha; \alpha) > 0$, $R(\beta; \beta) > 0$, $A(\alpha; \beta) > 0$ imply $A'(\alpha; \beta) > 0$; A' is ITP. □

We may now generalize Theorem G regarding Toda flow to ITP matrices in S_n:

Theorem 2.5. *If $A(0) \in S_n$ is ITP and $A(t)$ flows under the Toda flow $\dot{A} = AS - SA$; where $S = A^{+T} - A^{+}$, and A^{+} is the upper triangle of A, then $A(t)$ is ITP for all t.*

Proof. If $A(0)$ is ITP then it is NTN, so that, by Theorem G, $A(t)$ is NTN. $A(t)$ has the same staircase sequence for all t. Suppose $A(\alpha; \beta)$ is an extreme minor with $\alpha, \beta \in Q^0_{p,n}$ and $\alpha \geqslant \beta$. We may write

$$A(\alpha; \beta) = A(i, i+1, \ldots, i+p-1; j, j+1, \ldots, j+p-1)$$

and by a simple extension of the algebra in Gladwell [6] we may show that, under the Toda flow:

$$\dot{A}(\alpha; \beta) = \left(\sum_{k=1}^{i+p-1} a_{kk} - \sum_{k=1}^{j+p-1} a_{kk} \right) A(\alpha; \beta).$$

This equation for $A(\alpha; \beta)$ has the form

$$\dot{y}(t) = g(t)y(t)$$

and $g(t)$ is bounded by $|g(t)| \leqslant tr(A(t)) = tr(A(0))$. Thus the extreme minors retain the sign that they had at $t = 0$; $A(t)$ is TN and its extreme minors are positive; A is ITP. □

3. Reduction of ITP matrices to band form

Rainey and Halbetler [9] presented a constructive procedure that reduces a NTN matrix A to tridiagonal form $T : T = SAS^{-1}$; and they showed that S may be found so that it too is NTN. Cryer [2] independently proved the existence of such an NTN matrix S.

Suppose $A \in M_n$ is ITP with staircase sequences ρ, γ. If ℓ and u are fixed integers between 1 and n, and $\gamma(i) = \min(i + \ell, n)$, $\rho(i) = \min(i + u, n)$, $i = 1, 2, \ldots, n$, then A is said to have to *lower band width* ℓ and *upper band width* u; we say A has (ℓ, u) band form.

Suppose $A \in M_n$ is ITP with staircase sequences ρ, γ. If $\gamma(i) \geqslant \min(i + \ell, n)$, $\rho(i) \geqslant \min(i + u, n)$, $i = 1, 2, \ldots, n$ with at least one inequality being strict, then the analysis of Gasca and Peña [8] shows that A may be reduced to (ℓ, u) band form by using Neville elimination, and the reduced matrix will be ITP at each stage.

The reduction may be carried out in various ways, but for analytical purposes it is simplest to consider just the reduction of one half, the upper half, of the matrix. Consider a generic stage of the reduction. The first $r - 1$ rows have been reduced and the rth row has been partially reduced. Rename this partially reduced matrix A and assume that it is ITP; we show that the matrix at the next stage is also ITP.

The last non-zero (positive) entry in the $(r - 1)$th row is $a_{r-1,r-1+u}$. The next step is to eliminate the positive entry $r, p + 1$, where $p \geqslant r + u$. We take $B = AS$, $C = S^{-1}B$, where

$$S = \text{diag}\,(I_{p-1}, R, I_{n-p-1}), \tag{3.1}$$

where

$$R = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}. \tag{3.2}$$

The postmultiplication changes just column $p + 1$:

$$b_{i,p+1} = -ba_{i,p} + a_{i,p+1}, \quad i = 1, 2, \ldots, n.$$

Since $a_{i,j} = 0$ when $i = 1, 2, \ldots, r - 1,\ j \geqslant p$, the first $r - 1$ rows are unchanged by the postmultiplication. Since $a_{r,p+1} > 0$ and A is ITP, $a_{r,p} > 0$; take

$$b = a_{r,p+1}/a_{r,p}, \tag{3.3}$$

so that $b_{r,p+1} = 0$. The analysis of [8] shows that B is ITP.

We may also use the analysis of Section 4 of [8] to justify the factorization of an ITP band matrix. Since, according to Theorem 2.3, such a matrix may be factorized as $A = LU$, with L, U being ITP, it is sufficient to state the result for an upper triangular ITP band matrix

Theorem 3.1. *If $A_u \in M_n$ is an upper triangular ITP band matrix with upper band width u, then A_u may be factorized as*

$$A_u = D_0 D_1^{-1} E_1 D_1 D_2^{-1} E_2 \cdots D_{u-1} D_u^{-1} E_u D_u, \tag{3.4}$$

where $D_0 D_1, \ldots, D_u$ are diagonal matrices with positive entries and

$$D_u = diag(1, \dots, 1, d_{u,u+1}, \dots, d_{u,n}),$$

$$E_i = \begin{bmatrix} I_{i-1} & 0 \\ 0 & J_{n-i+1} \end{bmatrix}, \quad J_{n-i+1} = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}. \tag{3.5}$$

4. Filling in

Now consider filling in a ITP matrix so that it is TP. We consider filling in the lower half of the matrix so that it is full. Consider a generic stage of the filling in: the last $n - r$ columns have been filled in and column r has been partially filled in. Rename this partially reduced matrix A and assume that it is ITP; we show that the matrix at the next stage is also ITP.

The next step is to fill in the entry $p + 1, r$ where $r + 1 < p \leqslant n - 1$. We take $B = AS$, $C = S^{-1}B$, where

$$S = \text{diag}\,(I_{p-1}, R, I_{n-p-1}), \tag{4.1}$$

$$R = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}. \tag{4.2}$$

The postmultiplication is the critical step; it changes just column p

$$b_{ip} = a_{ip} - ba_{i,p+1}, \quad i = 1, 2, \dots, n. \tag{4.3}$$

The filling-in differs from the reduction in one fundamental respect: in the reduction, the matrix R in (4.1) is unique; b is given by (4.3). In the filling-in process, the matrix R is not unique; there are many values of b that will maintain the necessary ITP properties. Eq. (4.3) shows that b must be taken small enough positive so that all the inner minors of B are positive. It is the passage from A to B that is critical; the postmultiplication by S does not change the staircase sequences; B has the same sequences as A; if B is ITP then so is C, and it has just one extra non-zero, positive, entry. The important question is this: *which is the critical inner minor of* B, *the inner minor with the property that if it is positive then all the inner minors of B are positive*? We identify this critical minor in

Theorem 4.1. *Suppose* $A \in M_n$ *is ITP with* $\gamma(j) = n$ *for* $j \geqslant r + 1$ *and* $\gamma(r) = p$ *with* $r + 1 \leqslant p \leqslant n - 1$. *Define S by* $n - s = p - r$. *If*

$$B(s, \dots, n; r, \dots, p) > 0 \quad \textit{when } s \leqslant p \tag{4.4}$$

or

$$B(s + 1, \dots, n; r + 1, \dots, p) > 0 \quad \textit{when } s \geqslant p \tag{4.5}$$

then B is ITP.

Note that if $p = s$ then

$$B(s, \dots, n; r, \dots, p) = a_{sr} B(s + 1, \dots, n; r + 1, \dots, p)$$

so that (4.4) is equivalent to (4.5). The theorem identifies the critical minor as the extreme minor of maximal order, in the lower left hand corner of B, and that involves consecutive columns up to and including column p.

We need to establish three results.

Lemma 1. *Suppose an inner minor $B(i, \dots, m; j, \dots, p)$ is positive, then all inner minors $B(i - k + 1, \dots, m - k + 1; j, \dots, p)$ above it, are positive.*

Proof. We use induction on k. Let $P(k)$ be the statement '$B(i - k + 1, \dots, m - k + 1; j, \dots, p) > 0$'. Thus $P(1)$ is true. Suppose $P(k)$ is true for some k. Put $\alpha = \{i - k + 1, \dots, m - k\}$, $\beta = \{j, \dots, p - 1\}$. $P(k)$ states that

$$B(k) := B(\alpha \cup \{m - k + 1\}; \beta \cup \{p\}) > 0.$$

But

$$\begin{aligned} B(k) &= A(\alpha \cup \{m - k + 1\}; \beta \cup \{p\}) - bA(\alpha \cup \{m - k + 1\}; \beta \cup \{p + 1\}) \\ &= A_1(k) - bA_2(k) > 0. \end{aligned}$$

Suppose $B(k)$ is an inner minor of B, then $A_1(k)$ is an inner minor of A, and so positive. $A_2(k)$ may or may not be an inner minor of A. It is not an inner minor iff $p + 1 > \rho(m - k + 1)$; in that case $p + 1 > \rho(m - k' + 1)$ for $k' \geqslant k$, so that $A_2(k') = 0$, $B(k') = A_1(k') > 0$ for all inner minors with $k' \geqslant k$.

Otherwise, $p + 1 \leqslant \rho(m - k + 1)$; $A_2(k)$ is an inner minor, and so positive. Put

$$e_{ij} = A(\alpha \cup \{i\}; \beta \cup \{j\})$$

then $B(k) > 0$ states that

$$e_{m-k+1,p} - be_{m-k+1,p+1} > 0.$$

$P(k + 1)$ is the statement that $B(k + 1) > 0$:

$$B(k + 1) = A_1(k + 1) - bA_2(k + 1) = e_{i-k,p} - be_{i-k,p+1} \tag{4.6}$$

$A_1(k + 1)$ being an inner minor of A, is positive. If $p + 1 > \rho(m - k)$, then $A_2(k + 1)$ is not an inner minor, $B(k + 1) = A_1(k + 1) > 0$. Thus $P(k + 1)$ is true. If $p + 1 \leqslant \rho(m - k)$, then $A_2(k + 1)$ is an inner minor of A, so that (4.4) and (4.5) yield

$$A_2(k)B(k + 1) > A_1(k + 1)A_2(k) - A_1(k)A_2(k + 1) = G$$

and

$$G = e_{i-k,p}e_{m-k+1,p+1} - e_{i-k,p+1}e_{m-k+1,p}.$$

Sylvester's theorem states that

$$G = A(\alpha; \beta)A(\alpha \cup \{i - k, m - k + 1\}; \beta \cup \{p, p + 1\}). \tag{4.7}$$

We need to check that the second minor on the right is an inner minor of A. By hypothesis, both $B(k)$ and $B(k+1)$ are inner minors of B; their row and column numbers are as follows

	$B(k)$			$B(k+1)$			
row:	$i-k+1,$	$\dots$	$m-k+1$	$i-k,$	$i-k+1,$	$\dots$	$m-k$
column:	$j,$	$\dots$	p	$j,$	$j+1,$	$\dots$	$p.$

The row and column numbers of the second minor in (4.6) are

row:	$i-k,$	$i-k+1,$	$\dots$	$m-k,$	$m-k+1$
column:	$j,$	$j+1,$	$\dots$	$p,$	$p+1$

We need to check that $p+1 \leqslant \rho(m-k+1)$ and $m-k+1 \leqslant \gamma(p+1)$. The first follows from $p+1 \leqslant \rho(m-k)$; the second from the last entries of $B(k) : m-k+1 \leqslant \gamma(p)$. Thus $G > 0$, and $P(k+1)$ is true, and the lemma is established. □

Lemma 2. *Suppose an inner minor* $B(1) = B(i, \dots, m; j, \dots, p)$ *is positive, then all minors*

$$B(k) = B(i+k-1, \dots, m; j+k-1, \dots, p), \quad k = 1, 2, \dots, p-j+1$$

are positive. (*When* $k = p-j+1$, $j+k-1 = p$.)

Proof. Now the order of the minor decreases with k, until it is 1 when $k = p-j+1$. We note that if $B(1)$ is an inner minor of B then so are $B(k), k = 1, 2, \dots, p-j+1$.

We argue as in Lemma 1. Let $Q(k)$ be the statement '$B(k) > 0$'. Thus $Q(1)$ is true. Suppose $Q(k)$ is true for some k, $1 \leqslant k < p-j+1$. Put $\alpha = \alpha(k) = \{i+k-1, \dots, m\}$, $\omega = \{i+k, \dots, m\}$, $\tau = \{j+k, \dots, p-1\}$ then $Q(k)$ states that

$$\begin{aligned} B(k) &= B(\alpha; \tau \cup \{j+k-1, p\}) \\ &= A(\alpha; \tau \cup \{j+k-1, p\}) - bA(\alpha; \tau \cup \{j+k-1, p+1\}) \\ &= A_1(k) - bA_2(k) > 0. \end{aligned} \tag{4.8}$$

Since, by the hypothesis, $B(k)$ is an inner minor of B, $A_1(k)$ is an inner minor of A, and so positive. If $p+1 > \rho(m)$ then $A_2(k)$ is an outer minor, and so zero; in this case $A_2(k)$ is zero for all $k' \geqslant k$, so that all the inner minors $B(k')$ are positive for $k' > k$.

If $p+1 \leqslant \rho(m)$ then $A_2(k) > 0$. The statement Q_{k+1} concerns the positivity of

$$\begin{aligned} B(k+1) &= A(\omega; \tau \cup \{p\}) - bA(\omega; \tau \cup \{p+1\}) \\ &= A_1(k+1) - bA_2(k+1). \end{aligned} \tag{4.9}$$

Again, since $B(k+1)$ is an inner minor of B, $A_1(k+1)$ is an inner minor of A and so positive. If $p+1 > \rho(m)$, then $A_2(k+1) = 0$, and $B(k+1) = A_1(k+1) > 0$. Otherwise, $p+1 \leqslant \rho(m)$ and (4.7) and (4.8) imply

$$A_2(k)B(k+1) > A_1(k+1)A_2(k) - A_2(k+1)A_1(k) = H$$

and

$$H = A(\omega; \tau \cup \{p\})A(\omega; \tau \cup \{j+k-1, p+1\})$$
$$-A(\omega; \tau \cup \{p+1\})A(\alpha; \tau \cup \{j+k-1, p\})$$

and (2.2) states that

$$H = A(\omega; \tau \cup \{j+k-1\})A(\alpha; \tau \cup \{p, p+1\}).$$

We may verify by inspecting their row and column sequences that both these minors are inner minors of A, and so positive. Hence $Q(k+1)$ is true, and the lemma is established. □

Proof of Theorem 4.1. The minors of B fall into three groups: those that do not involve column p, those that involve column p and column $p+1$, those that involve column p but not $p+1$.

(i) Not column p. These are unchanged, so that $A(\alpha; \beta) \geqslant 0$ implies $B(\alpha; \beta) \geqslant 0$. If $B(\alpha; \beta)$ is an inner minor of B then $A(\alpha; \beta)$ is an inner minor of A, and so positive.

(ii) Both p and $p+1$. If $\{p, p+1\} \in \beta$, then

$$B(\alpha; \beta) = A(\alpha; \beta).$$

The conclusion is precisely as in (i).

(iii) p but not $p+1$.

We will use Theorem 2.1 to prove that B is ITP; this means that we need consider only inner minors $B(\alpha; \beta)$ with $\alpha, \beta \in Q^0_{n,k}$. We need to show that all such minors on columns $j, j+1, \ldots, p$ $(j \leqslant p)$ are positive. Lemma 1 shows that it is sufficient to show that the lowest inner minor on columns $j, j+1, \ldots, p$ is positive. First suppose s, defined by $n - s = p - r$ satisfies $s > p$. The hypothesis of the theorem is that the lowest inner minor on columns $r+1, \ldots, p$ is positive. Lemma 2 states that, in this case, the lowest inner minor on columns $j, \ldots, p$ is positive for $j \geqslant r+1$. It remains to show that the lowest inner minor is positive for $1 \leqslant j \leqslant r$. We prove this by induction. Suppose the lowest inner minor on columns $j, \ldots, p$ is positive for some $j \leqslant r+1$. Consider the lowest inner minor on columns $j-1, \ldots, p$. If $q = \gamma(j-1) < \gamma(j)$ then this minor is

$$B(q, q+1, \ldots, k; j-1, j, \ldots, p) = a_{q,j-1}B(q+1, \ldots, k; j, \ldots, p) > 0,$$

where $k - q = p - j + 1$. If $q = \gamma(j-1) = \gamma(j) = \cdots = \gamma(j+t-1) < \gamma(j+1)$ then the lowest inner minor on columns $j-1, \ldots, p$ is

$$B(q-t, q-t+1, \ldots, k-t; j-1, \ldots, p)$$
$$= A(q-t, \ldots, q; j-1, \ldots, j+t-1) *$$
$$B(q+1, \ldots, k-t; j+t, \ldots, p) > 0.$$

Thus, since the result is true for $j = r+1$, it holds for all $j = 1, 2, \ldots, r+1$. This concludes the proof that all inner minors in group (iii) are positive. Theorem 2.1 now states that B is ITP.

The case $s > p$ may be considered in a similar way with insignificant changes in the argument. □

We may show in a similar fashion that $C = S^{-1}B$ is ITP also.

Having established that the matrix remains ITP as one entry is filled in, we may conclude that the lower half, and also the upper half may be filled in until the matrix is full and thus TP.

5. Comments and applications

Our interest in total positivity stems from the fact that such matrices, particularly O, SO and ITP matrices appear in problems related to the vibration of discrete conservative systems, see Gantmacher and Krein [3], and Gladwell [10]. We have a particular interest in inverse eigenvalue problems for such systems; we give some background to this study.

We denote the spectrum of $A \in M_n$ by $\sigma(A)$. If $A \in M_n$ is O or TP then Perron's theorem on positive matrices, and the Cauchy–Binet theorem together imply that

$$\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \quad \text{where } 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n.$$

(Note that this holds for $A \in M_n$, even though, in applications, A is usually symmetric.) We denote the spectrum of the matrix $A[r+1, \ldots, n]$ by $\sigma_r(A)$, $r = 1, 2, \ldots, n-1$, and write $\sigma(A) = \sigma_0(A)$. If $A \in M_n$ is O or TP, then $\sigma_r(A)$ is a set of $n-r$ discrete positive numbers, and $\sigma_r(A)$, $\sigma_{r-1}(A)$ strictly interlace. In particular,

$$\sigma_1(A) = \{\mu_1, \mu_2, \ldots, \mu_n\}, \text{ where } 0 < \lambda_1 < \mu_1 < \mu_2 < \cdots < \mu_{N-1} < \lambda_n.$$

The classical result on the construction of oscillatory matrices with given spectra is that there is a unique symmetric oscillatory tridiagonal matrix A with given strictly interlacing spectra $\sigma(A)$, $\sigma_1(A)$. There are many ways to find A, for which see the review article by Boley and Golub [11] or Gladwell [10].

There is no straightforward generalization of this result to band matrices. Boley and Golub showed how to construct a symmetric band matrix with half band width p from $p+1$ spectra $\sigma_0, \sigma_1, \ldots, \sigma_p$. Now, however, if $p > 1$, the strict interlacing of σ_r and σ_{r-1} for $r = 1, \ldots, p$, while being necessary, is not sufficient for A to be oscillatory, as shown by the following counterexample:

$$n = 3, \quad p = 2, \quad \sigma = \{1, 4, 6\}, \quad \sigma_1 = \{2, 5\}, \quad \sigma_2 = 3.$$

Clearly $a_{33} = 3, \quad a_{22} = 4, \quad a_{11} = 4, \quad a_{23}a_{32} = 2$, so that

$$A = \begin{bmatrix} 4 & a_{12} & a_{13} \\ a_{21} & 4 & a_{23} \\ a_{31} & a_{32} & 3 \end{bmatrix}.$$

Using the products of eigenvalues we find

$$a_{12}a_{21} + a_{31}a_{13} = 4$$
$$48 + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - 4a_{31}a_{13} - 3a_{12}a_{21} - 4a_{23}a_{32} = 24$$

giving

$$a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + a_{12}a_{21} = 0$$

which clearly has no strictly positive solution, symmetric or not.

Gladwell [12], in the context of inverse eigenvalue problems for a vibrating beam in flexure, encountered a symmetric oscillatory pentadiagonal matrix of the form

$$A = X_0 E_1 X_1 E_1 X_2^2 E_1^{\mathrm{T}} X_1 E_1^{\mathrm{T}} X_0 \tag{5.1}$$

where X_0, X_1, X_2 are diagonal matrices with positive entries, and E_1, is given in (3.5). One question immediately arises. Can *any* symmetric pentadiagonal ITP matrix be written in this form? The matrix A in (5.1) may be factorized as

$$A = (X_0 E_1 X_1 E_1 X_2)(X_2 E_1^{\mathrm{T}} X_1 E_1^{\mathrm{T}} X_0) = UL.$$

Thus, the question is equivalent to this: can any upper triangular ITP matrix with band width 2 be written as

$$U = X_0 E_1 X_1 E_1 X_2? \tag{5.2}$$

We showed in Theorem 3.1 that any such matrix may be written as

$$U = D_0 D_1^{-1} E_1 D_1 D_2^{-1} E_2 D_2 \tag{5.3}$$

with positive diagonal D_0, D_1, D_2. Is the converse true? That is, given U of the form (5.3), can we write it as (5.2) for positive X_0, X_1, X_2? If

$$D_0 D_1^{-1} E_1 D_1 D_2^{-1} E_2 D_2 = X_0 E_1 X_1 E_1 X_2$$

then

$$E_1 (D_1 D_2^{-1}) E_2 = (D_1 D_0^{-1} X_0) E_1 X_1 E_1 (X_2 D_2^{-1}).$$

This is equivalent to

$$E_1 B E_2 = Y_0 E_1 X_1 E_1 Y_2.$$

If $B = \operatorname{diag}(b_1, b_2, \ldots, b_n)$, $X_1 = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ then after some routine algebra we reach the equations

$$\frac{(x_1 + x_2)(x_2 + x_3)}{x_2^2} = \frac{b_2 + b_3}{b_2} \tag{5.4}$$

$$\frac{(x_{i-1}+x_i)(x_i+x_{i+1})}{x_i^2} = \frac{(b_{i-1}+b_i)(b_i+b_{i+1})}{b_i^2}, \quad i = 3, 4, \ldots, n-1. \tag{5.5}$$

It is possible to solve these explicitly for $x_1, \ldots, x_n$, but we are concerned to show only that they do have a positive solution. Clearly, they have the solution $x_1 = 0, x_i = b_i, i = 2, \ldots, n$, but this is not acceptable. Normalize the x_i with respect to x_2, i.e., put $y_1 = x_i/x_2, i = 1, \ldots, n$, then (5.4) becomes

$$(y_1 + 1)(1 + y_3) = (b_2 + b_3)/b_2$$

so that

$$y_1 = \frac{(b_3/b_2) - y_3}{1 + y_3}. \tag{5.6}$$

Eq. (5.5) becomes

$$\frac{(y_{i-1}+y_i)(y_i+y_{i+1})}{y_i^2} = \frac{(b_{i-1}+b_i)(b_i+b_{i+1})}{b_i^2}, \quad i = 3, \ldots, n-1. \tag{5.7}$$

Given y_3, we can solve equation (5.7) sequentially for $y_4, \ldots, y_{n-1}$, and when $y_3 = b_3/b_2$ we have $y_i = b_i/b_2, i = 4, \ldots, n$. Since, clearly, the y_i are continuous functions of y_3 near b_3/b_2, we can find an interval $((b_3/b_2) - \varepsilon, b_3/b_2)$ in which $y_4, \ldots, y_n$ will be positive, and, as (5.6) shows, y_1 will also be positive: the equations have a positive solution: any symmetric ITP pentadiagonal matrix may be written in the form (5.1).

In a forthcoming paper we show how this result may be used in conjunction with the results of Gladwell [12] to find the necessary and sufficient conditions for three spectra $\sigma_0, \sigma_1, \sigma_2$ to correspond to a symmetric ITP pentadiagonal matrix.

However, while it is not possible to construct a symmetric ITP pentadiagonal matrix having any *three* given strictly interlacing spectra, the analysis given in Sections 3, 4 allows us to construct a family of such matrices with any two strictly interlacing positive spectra σ, σ_1. We first construct the unique symmetric tridiagonal matrix with spectra σ, σ_1, fill it out to TP form and then reduce it to pentadiagonal form. All the operations carried out will involve pre- and postmultiplication by matrices of the form (4.1) with $p \geqslant 2$, and such operations will not change σ or σ_1.

6. Conclusions

In the author's opinion, the concept of an ITP i.e., NASTP, matrix brings a clearer understanding and a unification to the field of total positivity. As shown by the theorems in Section 2, an ITP matrix has properties that are closer to those of a TP matrix than to those of a matrix that is merely NTN, and it is ITP, rather than TP or NTN matrices that often appear in applications.

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