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GENERIC ELEMENT MATRICES SUITABLE FOR FINITE ELEMENT MODEL UPDATING

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Updating procedures are concerned with the construction of mass, stiffness and (possibly) damping matrices, near to some base matrices, which yield some response data close to some experimental values. One of the fundamental questions in updating relates to the criteria for allowable mass and stiffness matrices. In practice the updated matrices are often not physically meaningful. In this paper we start from the element level and consider what criteria must be satisfied by the element mass and stiffness matrices, and thus what parameters there are which can be optimised in the updating. We show that the recognised matrices appearing in special finite element formulations are members of families which may be obtained, in a logical manner, by applying these criteria.

1. INTRODUCTION

This paper is concerned with finite element modeling. The term modeling denotes the process of setting up a set of equations which can be used to predict the behaviour of a system. The behaviour of interest to us is vibration, and the modeling is to be carried out using some finite element model. In this paper we are concerned exclusively with the small undamped vibration of a conservative system; such systems form the basis, i.e. the starting point, for most research in the field. We must make a distinction between two kinds of problems involving finite element modeling. In the first, so-called direct problems, we are presented with a (mechanical) system and asked to model it. We idealise the parts of the system as beams, plates, shells, joints, etc. and construct the element matrices for each part by using the classical theories which provide expressions for the kinetic and potential energies of the elements. Once we have the mass and stiffness matrices for the elements we can assemble them by using the established procedures, and analyse the whole. But there is a second class of problems, model updating problems, and it is with these that we are concerned. In model updating we are not dealing just with the direct problem. We do have a system, and we have modeled it, but our predictions do not agree with experiment. In effect what we want to do now is to change the model, a little, so that it will model the behaviour of the system. This second, updating, stage is thus a system (or model) identification problem: what system behaves in the way exhibited by our experiments? Of course our problem is not an identification problem in the widest sense; we have a model which predicts results near, in some sense, to the experimental values; we just need to fine-tune it. There are essentially two reasons why our model fails to predict correctly, and correspondingly these are two kinds of model identification problem, which we will label Type A and Type B; Type A problems are much more difficult than Type B. To make the distinction we notice that the process of setting up a finite element model has two parts: including all the various effects due, for example, to flexure, shear, Poisson's ratio, coupling of flexure and torsion, joint flexibility,

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etc. which affect the behavior of the system; assigning the numerical values of all the parameters, such as lengths, thicknesses, Poisson's ratios, coupling factors, which quantify these effects, and so determine the model. We may now define the two types of problem.

Type A. The initial model has neglected important effects. In the identification problem we must identify and re-introduce the relevant effects, and choose the associated parameters to give predictions agreeing with experiment.

Type B. The initial model has included all the relevant effects. The identification problem is merely that of finding the correct numerical values of the model parameters.

Let us be more specific. Our system is specified by a mass matrix M and stiffness matrix K; M and K are symmetric, M positive definite and K positive semi-definite. In Type B problems we have matrices M and K which are qualitatively correct, in the sense that they mirror all the effects which are in operation, this means in particular that they have the correct pattern of zero and non-zero coefficients, but the numerical values of their non-zero coefficients derived from the parameters defining the model, are incorrect. The problem of updating these parameter values to give results agreeing with experiment has been studied intensively. These problems are not (absolutely) easy, but are easy compared to problems of Type A.

In Type A problems the initial matrices M and K are not even qualitatively correct; effects have been neglected, there are terms which are zero which should not be, and the values of the terms which have been included are restricted by the inadequate expressions which have been used to construct them. Now the first fundamental question which must be asked is: what changes in M and K are allowable?

At this point it might be appropriate to review the literature on updating, but we will not; the literature is vast and there are a number of excellent recent reviews, see for example Natke [1], Imregun and Visser [2] Mottershead and Friswell [3]. Instead we shall briefly review some of the ideas that have been used.

Traditionally it has been found (or thought) that prediction errors have arisen from incorrect modeling of K, rather that M; in some of the early papers M was assumed to be correct, and only K was to be changed. What changes were allowed in K? Some researchers allowed any symmetrical changes. This was unsatisfactory because it could not be justified on physical grounds; the structure of the matrix might not bear any resemblance to the way in which the parts of the physical system were connected to each other. Moreover such changes might destroy the positive semi-definiteness of K. Some researchers allowed symmetrical changes which maintained the distribution of zero and non-zero elements, and allowed both M and K to vary in this way. Again this could destroy positive (semi-)definiteness. Some researchers effectively limited themselves to Type B problems. They realised that M and K are built up from element matrices, and these element matrices can often be written as sums of products of certain physical parameters (masses, lengths, stiffnesses, as appropriate) and certain numerical matrices arising, for example, from assumed shape function integrations. They therefore allowed the physical parameters to vary (but remain positive) and kept the numerical matrices fixed. This approach sometimes provides an acceptable compromise: the model may be fairly flexible, consistent with the structure of the physical system, and have the required positivity properties. But the method will fail if the problem is really of Type A, and there are effects which have been ignored; in attempting to obtain results agreeing with experiment with this procedure in such a situation, we find that certain of the parameters take on unrealistic values in an attempt to compensate for the parameters which have been ignored, and even then the model fails to predict all the test results.

These are some of the considerations which have prompted us to consider generic element mass and stiffness matrices. In essence our approach is as follows. An element is an object whose motion is defined by a certain number, r, of degrees of freedom. It will have a mass matrix, symmetric and positive definite and of order r. It will have a certain number, $d(\leqslant 6)$, rigid body modes, and will have a stiffness matrix, symmetric and positive semi-definite and of rank r–d. If the element possesses certain symmetry properties, then its matrices will exhibit these properties. This appears to be the sum total of requirements for the element matrices. This means that any particular matrices derived by considering certain effects in an element must fall into the general classes described above. However, in the updating procedure we do not know all the "effects" which must be considered; there may be some which have not been described in the existing literature. Therefore, instead of trying to incorporate all the possible effects into the matrices by introducing various physical parameters and finding the appropriate values of these parameters, we merely assume that the matrices for the element belong to the generic families appropriate to these elements, and find the parameters needed to specify members in those families. In this way we can take account not only of those effects we are aware of, but also all the possible effects which can be accommodated by the matrices in the families.

2. ELEMENT MASS AND STIFFNESS MATRICES

From now on we will discuss only element matrices; we will use the notation $\mathbf{M}$ for $\mathbf{M}^e$, $\mathbf{K}$ for $\mathbf{K}^e$. The free vibration of the element itself is governed by the equation

$$(\mathbf{K}-\lambda\mathbf{M})\phi=0. \tag{1}$$

If the element has r degrees of freedom and $d(\leqslant 6)$ rigid body modes, $(\phi_i)_1^d$, we write the $r\times r$ matrix $\mathbf{\Phi}$ as

$$\mathbf{\Phi}=[\phi_1,\ldots,\phi_d|\phi_{d+1},\ldots,\phi_r]=[\mathbf{\Phi}_R,\mathbf{\Phi}_S] \tag{2}$$

where R denotes rigid body, and S strain. If the modes are normalised w.r.t. $\mathbf{M}$ we find

$$\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}=m_0 I, \quad \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}=k_0\mathbf{\Gamma} \tag{3}$$

where

$$\mathbf{\Gamma}=\frac{m_0}{k_0}\,\mathrm{diag}(0,0,\ldots,0,\lambda_{d+1},\ldots,\lambda_r)=[\mathbf{0},\mathbf{\Gamma}_S], \tag{4}$$

m_0, k_0 are some standard mass and stiffness, and

$$\mathbf{K}\phi_i=0, \quad i=1,2,\ldots,d, \quad \text{i.e. } \mathbf{K}\mathbf{\Phi}_\mathbf{R}=0. \tag{5}$$

This analysis shows one way of constructing a family of $\mathbf{M}$, $\mathbf{K}$ matrices: we specify the d rigid-body modes in $\mathbf{\Phi}_R$, the positive dimensionless eigenvalues making up $\mathbf{\Gamma}_S$, and the remaining modes in $\mathbf{\Phi}_S$. Provided that $\mathbf{\Phi}$ is non-singular, we can find $\mathbf{M}$, $\mathbf{K}$ from equation (3), i.e.

$$\mathbf{M}=m_0\mathbf{\Phi}^{-T}\mathbf{\Phi}^{-1}, \quad \mathbf{K}=k_0\mathbf{\Phi}^{-T}\mathbf{\Gamma}\mathbf{\Phi}^{-1} \tag{6}$$

although we shall not in fact use these equations, which involve the inversion of $\mathbf{\Phi}$, directly. The $\mathbf{M}$ so constructed will be positive definite, $\mathbf{K}$ will be positive semi-definite and have d rigid body modes $\mathbf{\Phi}_R$. Note that we are still at a conceptual stage; the modes making up $\mathbf{\Phi}$ are modes of an element; they have nothing to do with measured modes.

Let $\mathbf{M}_0$, $\mathbf{K}_0$ be some datum pair of mass and stiffness matrices for an element, and suppose that the modes for that element are the columns of the matrix $\mathbf{\Phi}_0$. Let $\mathbf{M}$, $\mathbf{K}$ be any other member of the generic family, and let its modes be the columns of $\mathbf{\Phi}$. Since both $\mathbf{\Phi}$ and $\mathbf{\Phi}_0$ are non-singular there is a unique matrix $\mathbf{S}$ relating $\mathbf{\Phi}$, $\mathbf{\Phi}_0$ by the equation

$$\mathbf{\Phi} = \mathbf{\Phi}_0\mathbf{S}^{-1} \quad \text{or} \quad \mathbf{\Phi}_0 = \mathbf{\Phi}\mathbf{S}. \tag{7}$$

On physical grounds we now restrict $\mathbf{S}$ so that the number of rigid body modes remains the same, d, and the new rigid body modes are linear combinations of the original ones. Equation (7) now becomes

$$\mathbf{\Phi}_0 = [\mathbf{\Phi}_{0R}, \mathbf{\Phi}_{0S}] = [\mathbf{\Phi}_R, \mathbf{\Phi}_S]\begin{bmatrix} \mathbf{S}_R & \mathbf{S}_{RS} \\ \mathbf{0} & \mathbf{S}_S \end{bmatrix}. \tag{8}$$

This means that the new strain modes may be combinations of all the original modes. We may further restrict $\mathbf{S}$ using symmetry considerations. If the element is governed by some symmetry group, then its modes will reflect the properties of the group. If the new element retains this symmetry, then new modes with a particular symmetry will be linear combinations of the old modes with the same symmetry; this will produce diagonal blocks in $\mathbf{S}$. In particular if the new and old models have the same centre of mass and principal axes of inertia at the rigid body level, then $\mathbf{S}_R$ will be diagonal; if they have the same mass and moments of inertia, then $\mathbf{S}_R$ will be the unit matrix $\mathbf{I}_d$.

Inserting (7) into (6) we find

$$\mathbf{M} = m_0\mathbf{\Phi}_0^{-T}\mathbf{S}^T\mathbf{S}\mathbf{\Phi}_0^{-1}, \tag{9}$$

$$\mathbf{K} = k_0\mathbf{\Phi}_0^{-T}\mathbf{S}^T\mathbf{\Gamma}\mathbf{S}\mathbf{\Phi}_0^{-1}. \tag{10}$$

Now using the fact that $m_0\mathbf{\Phi}_0^{-T} = \mathbf{M}_0\mathbf{\Phi}_0$ we find

$$\mathbf{M} = m_0^{-1}\mathbf{M}_0\mathbf{\Phi}_0\mathbf{U}\mathbf{\Phi}_0^T\mathbf{M}_0, \tag{11}$$

$$\mathbf{K} = (k_0 m_0^{-2})\mathbf{M}_0\mathbf{\Phi}_{0S}\mathbf{V}\mathbf{\Phi}_{0S}^T\mathbf{M}_0, \tag{12}$$

where

$$\mathbf{U} = \mathbf{S}^T\mathbf{S}, \quad \mathbf{V} = \mathbf{S}_S^T\mathbf{\Gamma}_S\mathbf{S}_S. \tag{13}$$

Equations (11) and (12) show that the terms which may be updated appear in two-dimensionless symmetric positive definite matrices $\mathbf{U}$, $\mathbf{V}$ of order r and r–d respectively. Note that the new modes $\mathbf{\Phi}$ are still orthogonal with respect to the new mass and stiffness matrices $\mathbf{M}$, $\mathbf{K}$, i.e. equations (3) still hold.

Before going further, we make some comments. The most common procedure for constructing element mass and stiffness matrices involves assumed shape functions; the displacements in an element are expressed as linear combinations of these functions; after performing the necessary integrations over the element we obtain the kinetic and strain energies as quadratic forms in the generalised nodal displacements; the kernels of these forms provide the mass and stiffness matrices for the element. We could obtain a family of element matrices by taking different assumed modes. As the finite element method has evolved, rules have been formulated for the choice of assumed modes, e.g. the rigid body modes of the structure must be expressible as combinations of the assumed modes, but these rules still allow for a variety of acceptable assumed modes; the element matrices so formed with these various assumed modes would all fall into the generic families we have described. But the generic family is wider than this, because it is not based on particular integral expressions for the kinetic and strain energies for the element. Instead, the energies are expressed directly as quadratic forms in the generalised nodal displacements. This means that the family will include all the possible element matrices for which the energies may be so expressed. Thus the family will include all the matrices which may be obtained including all the various coupling and shear effects, providing the energies may be expressed in terms of generalised nodal displacements.

We have chosen to express the new matrices $\mathbf{M}$ and $\mathbf{K}$ in terms of a datum set $\mathbf{M}_0$, $\mathbf{K}_0$, because we are concerned with updating, i.e. fine-tuning a starting model. We could have proceeded in a more abstract way, simply writing down expressions for $\mathbf{M}$, $\mathbf{K}$ such that $\mathbf{M}$ is positive definite, $\mathbf{K}$ is positive semi-definite with d rigid body modes. This is done in Section 6 and is sometimes simpler than the procedure we have described, but is in fact no more general. We now consider some examples.

3. ROD ELEMENT

Consider a straight, thin rod of length L in longitudinal vibration. For a lumped mass model of a uniform element

$$\mathbf{M} = m_0 \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad \mathbf{K} = k_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{14}$$

where $m_0 = \rho AL$, $k_0 = EA/L$. There is one rigid body mode, one strain mode, and

$$\mathbf{\Phi}_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{15}$$

Now $\mathbf{S}$ is a 2×2 matrix which may be written

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_{12} \\ \mathbf{0} & \mathbf{S}_2 \end{bmatrix} \tag{16}$$

so that $\mathbf{V}$ is simply the scalar $\gamma_2 S_2^2$, and

$$\mathbf{K} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad k = k_0 \gamma_2 S_2^2/4. \tag{17}$$

We note that $\mathbf{K}$ does not change its form; it is specified by one parameter k. If γ_2 retains its old value $\gamma_{0,2} = 4$, then $k = k_0 S_2^2$; alternatively, if $k = k_0$, then $\gamma_2 = 4\gamma_{0,2} S_2^2$.

Now consider $\mathbf{M}$. If the element is symmetrical about its centre, then $S_{12} = 0$ and the product (11) becomes

$$\mathbf{M} = \begin{bmatrix} m_1 & m_{12} \\ m_{12} & m_1 \end{bmatrix} \tag{18}$$

where

$$m_1 = \frac{m_0}{4}(S_1^2 + S_2^2), \quad m_{12} = \frac{m_0}{4}(S_1^2 - S_2^2). \tag{19}$$

If the new element has the same mass as the old, so that $[1, 1]\mathbf{M}[1, 1]^T = m_0$, then $S_1 = 1$, and there is a single parameter, S_2 to be identified; different historical models may be associated with different values of S_2. Thus $S_2 = 1$ gives the original lumped mass model;

$S_2=\sqrt{3}/3$ gives the consistent mass model derived from linear shape functions; $S_2=\sqrt{2}/2$ gives the model considered by Stavrinidis [4] derived from shape functions

$$N_1=\cos^2\theta, \quad N_2=\sin^2\theta, \quad \text{where } \theta=\pi x/2L. \tag{20}$$

If we model a uniform continuous rod of length L by using n equal elements, take $k=k_0$, and arbitrary S_2, we find that the eigenvalues of the rod, when the left-hand end is fixed, are given by

$$\lambda_i=\frac{4E}{\rho L^2}\cdot\frac{n^2}{S_2^2+\cot^2(\theta_i/2n)}, \tag{21}$$

where $\theta_i=(2i-1)\pi/2$, $i=1, 2, \ldots, n$ for a cantilever; $\theta_i=i\pi$, $i=1, 2, \ldots, n-1$ for a fixed–fixed rod.

We compare this with the exact values

$$\lambda_i^*=\frac{4E}{\rho L^2}\cdot\theta_i^2, \tag{22}$$

by using the expansion

$$\cot^2 x=\frac{1}{x^2}-\frac{2}{3}-\frac{2x^2}{45}+\ldots, \quad |x|<\pi. \tag{23}$$

We find that for large n

$$\lambda_i=\lambda_i^*\left\{1+\frac{a\theta_i^2}{4n^2}+\left(a^2+\frac{2}{45}\right)\frac{\theta_i^4}{16n^4}+\ldots\right\}, \tag{24}$$

where $a=2/3-S_2^2$. This shows that the errors in λ_i are generally $O(n^{-2})$; λ_i will generally be less than λ_i^* when $a<0$, greater than λ_i^* when $a>0$; the former will occur in the lumped mass model and Stavrinidis' model. As MacNeal [8] has shown, the errors are $O(n^{-4})$ when $S_2^2=2/3$, i.e. $S_2=\sqrt{2/3}$.

We emphasise that comparing λ_i and λ_i^* is a matter which belongs to theoretical direct modeling of a uniform rod. (We discuss it only to back our claim that the various models that have been developed in the past all belong to our generic family.) Our concern is different. For us it is important to know that the most general model of the stiffness matrix of a rod element, under our assumptions, is equation (17); the most general model of the mass matrix of a symmetrical element is given by equations (18) and (19). Thus in updating, we have one parameter defining $\mathbf{K}$; two defining $\mathbf{M}$, and only one, S_2, if the mass of the element is known.

4. BEAM ELEMENTS

We start with a simple lumped-mass Euler–Bernoulli element with a stiffness matrix obtained by using the usual four cubic shape functions. For a uniform element this gives

$$\mathbf{M}_0=m_0\begin{bmatrix}\frac{1}{2} & & & \\ & \frac{1}{24} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{24}\end{bmatrix}, \quad \mathbf{K}_0=k_0\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix} \tag{25}$$

where $m_0 = A\rho L$, $k_0 = EI/L^3$ and the displacement vector is $\{w_{i-1}, Lw'_{i-1}, w_i, Lw'_i\}^T$. For this model

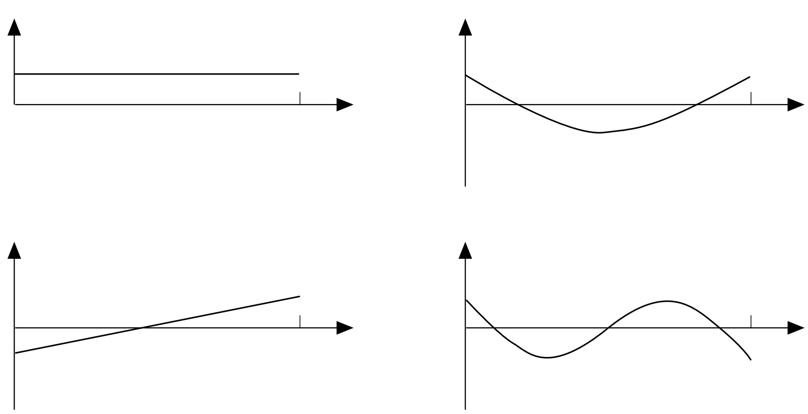
$$\boldsymbol{\Phi}_0 = \begin{bmatrix} 1 & -\frac{\alpha}{2} & 0 & \frac{1}{2} \\ 0 & \alpha & 2\alpha & 3 \\ 1 & \frac{\alpha}{2} & 0 & -\frac{1}{2} \\ 0 & \alpha & -2\alpha & 3 \end{bmatrix}, \quad \boldsymbol{\Gamma}_0 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 48 & \\ & & & 192 \end{bmatrix} \tag{26}$$

and we have used the abbreviation $\alpha = \sqrt{3}$. Now $r=4$, $d=2$; there are two rigid body modes, two strain modes, as shown in Fig. 1; one of each kind is symmetric, the other antisymmetric, about the centre of the element. If the symmetry of the element is preserved, then $\mathbf{S}$ will have the form

$$\mathbf{S} = \begin{bmatrix} S_1 & 0 & S_{13} & 0 \\ & S_2 & 0 & S_{24} \\ & & S_3 & 0 \\ & & & S_4 \end{bmatrix} \tag{27}$$

so that each matrix $\mathbf{S}_R$, $\mathbf{S}_{RS}$, $\mathbf{S}_S$ will be diagonal. This means that the matrix $\mathbf{V}$ will be diagonal, i.e. $\mathbf{V} = \text{diag}\,\{v_1, v_2\}$, so that equation (12) gives the most general stiffness matrix as

$$\mathbf{K} = k_0 \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{\alpha}{12} & \frac{1}{8} \\ 0 & -\frac{1}{4} \\ -\frac{\alpha}{12} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} v_1 & \\ & v_2 \end{bmatrix} \begin{bmatrix} 0 & \frac{\alpha}{12} & 0 & -\frac{\alpha}{12} \\ \frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \end{bmatrix} \tag{28}$$



Rigid body modes Strain modes

Figure 1. Beam element having two symmetrical and two antisymmetrical modes.

where $v_1 = \gamma_3 S_3^2$, $v_2 = \gamma_4 S_4^2$. There are just two parameters which can be varied; in our initial model they have the values

$$v_1 = 48, \quad v_2 = 192. \tag{29}$$

The Timoshenko beam element belongs to this family and is obtained by setting

$$v_1 = 48, \quad v_2 = 192/(1+12\beta) \tag{30}$$

where

$$\beta = \frac{CEI}{AL^2G} \tag{31}$$

and C is a shape factor. This $\mathbf{K}$ has the familiar form

$$\mathbf{K} = \frac{k_0}{1+12\beta}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4+12\beta & -6 & 2-12\beta \\ -12 & -6 & 12 & -6 \\ 6 & 2-12\beta & -6 & 4+12\beta \end{bmatrix}. \tag{32}$$

Remember that in Timoshenko beam theory, plane sections do not remain normal to the neutral axis; there are two quantities w and ψ at each point. The stiffness and mass matrices are changed to take into account of the fact that $\psi \neq w'$, but the degrees of freedom for an element remain

$$w_{i-1}, \quad Lw'_{i-1}, \quad w_i, \quad Lw'_i.$$

In order to avoid shear locking, Hughes derives a stiffness matrix of the beam element as the sum of two parts: a bending stiffness based on shape functions and a shear stiffness based on one- or two-point Gaussian quadrature; thus

$$\mathbf{K}^{(i)} = \mathbf{K}_b + \mathbf{K}_S^{(i)}, \quad i = 1, 2. \tag{33}$$

It may be readily verified that $\mathbf{K}^{(1)}$ is the member of the family with

$$v_1 = 48, \quad v_2 = 16\kappa, \quad \kappa = Al^2G/EI. \tag{34}$$

The suggestion by MacNeal [9] that κ be replaced by

$$\kappa^* = \frac{\kappa}{1+\kappa/12} \tag{35}$$

reproduces the Timoshenko beam element with $C = 1$. The matrix $\mathbf{K}^{(2)}$ corresponds to

$$v_1 = 48 + 4\kappa, \quad v_2 = 16\kappa. \tag{36}$$

Tessler and Dong [10] provided a historical commentary on the various Timoshenko-like finite element models. Our analysis shows that all the various 2 node, 4 dof stiffness matrices which have been suggested are all particular cases of that given in equation (28); in updating we have two parameters v_1, v_2 to vary.

Now consider the possible mass matrices which may arise from the $\mathbf{S}$ in equation (27). $\mathbf{U}$ will have the form

$$\mathbf{U} = \mathbf{S}^T\mathbf{S} = \begin{bmatrix} S_1^2 & 0 & S_1S_{13} & 0 \\ 0 & S_2^2 & 0 & S_2S_{24} \\ S_1S_{13} & 0 & S_{13}^2 + S_3^2 & 0 \\ 0 & S_2S_{24} & 0 & S_{24}^2 + S_4^2 \end{bmatrix} = \begin{bmatrix} u_1 & 0 & u_{13} & 0 \\ 0 & u_2 & 0 & u_{24} \\ u_{13} & 0 & u_3 & 0 \\ 0 & u_{24} & 0 & u_4 \end{bmatrix}. \tag{37}$$

The moment of inertia of a uniform beam of mass m_0 and length L about its centre is $m_0L^2/12$. The original model was obtained by simply dividing this mass and moment of inertia equally between the two ends. If we require that the new model represent a beam with mass m_0 and moment of inertia $m_0L^2/12$, we must constrain $\mathbf{M}$ so that

$$[1, 0, 1, 0]\mathbf{M}[1, 0, 1, 0]^T = m_0, \tag{38}$$

$$\left[-\frac{L}{2}, L, \frac{L}{2}, L\right]\mathbf{M}\left[-\frac{L}{2}, L, \frac{L}{2}, L\right]^T = \frac{m_0L^2}{12} \tag{39}$$

which give $S_1 = 1$, $S_2 = 1/2$.

Again we can identify various choices which have been made in the past. We can find the matrix consistent with the four cubic shape functions by choosing the values

$$S_{13} = \frac{\alpha}{3}, \quad S_{24} = -\frac{3\alpha}{10}, \quad S_3 = \frac{1}{\sqrt{15}}, \quad S_4 = \frac{2}{5\sqrt{7}}. \tag{40}$$

Archer [11] derived a consistent mass matrix for the Timoshenko element, and this corresponds to the values

$$S_{13} = \frac{\alpha}{3}, \quad S_{24} = -\frac{\alpha(3+20\beta)}{10(1+12\beta)}, \quad S_3 = \frac{1}{\sqrt{15}}, \quad S_4 = \frac{2}{5\sqrt{7}(1+12\beta)}. \tag{41}$$

[It is amazing to realise that the cumbersome expressions in [11] can be derived simply by changing S_{24} and S_4 from those given in equation (40)]. The eigenvalues derived from the consistent mass and stiffness model of a uniform beam generally have a discretisation error which is $O(n^{-4})$. The parameter values

$$S_{13} = \frac{\alpha}{3}, \quad S_{24} = -\frac{3\alpha}{10}, \quad S_3 = \frac{1}{\sqrt{15}}, \quad S_4 = \frac{2\sqrt{38}}{5\sqrt{21}}. \tag{42}$$

give a mass matrix

$$\mathbf{M} = \frac{m_0}{840}\begin{bmatrix} 326 & 51 & 94 & -19 \\ 51 & 15 & 19 & -6 \\ 94 & 19 & 326 & -51 \\ -19 & -6 & -51 & 15 \end{bmatrix} \tag{43}$$

which, with the stiffness matrix in equation (25) yields a discretisation error which is $O(n^{-6})$, as shown by Stavrinidis *et al*. [12]. [Maybe one could obtain an $O(n^{-8})$ error by taking $\mathbf{K}$ to be another member of the family given by equation (28).]

5. IN-PLANE FRAME ELEMENT

A plane frame element is a combination of a beam in flexure and a rod in torsion. In the usual notation we assume that the element has length L, bending stiffness k_0, mass m_0. The torsional effects are introduced through dimensionless parameters

$$\tau = \frac{GJ}{EI}, \quad \sigma = 140\,\frac{J}{AL^2} \tag{44}$$

where J is the axial moment of inertia. With the co-ordinates

$$w_1, \quad Lw_1', \quad L\theta_1, \quad w_2, \quad Lw_2', \quad L\theta_2 \tag{45}$$

the usual consistent stiffness and mass matrices are

$$\mathbf{K}_0 = k_0 \begin{bmatrix} 12 & 6 & 0 & -12 & 6 & 0 \\ 6 & 4 & 0 & -6 & 2 & 0 \\ 0 & 0 & \tau & 0 & 0 & -\tau \\ -12 & -6 & 0 & 12 & -6 & 0 \\ 6 & 2 & 0 & -6 & 4 & 0 \\ 0 & 0 & -\tau & 0 & 0 & \tau \end{bmatrix} \tag{46}$$

$$\mathbf{M}_0 = \frac{m_0}{420} \begin{bmatrix} 156 & 22 & 0 & 54 & -13 & 0 \\ 22 & 4 & 0 & 13 & -3 & 0 \\ 0 & 0 & \sigma & 0 & 0 & \dfrac{\sigma}{2} \\ 54 & 13 & 0 & 156 & -22 & 0 \\ -13 & -3 & 0 & -22 & 4 & 0 \\ 0 & 0 & \dfrac{\sigma}{2} & 0 & 0 & \sigma \end{bmatrix}. \tag{47}$$

These matrices, which are derived on the basis of no coupling between bending and twisting have the following eigenvalues and eigenvectors

$$\mathbf{\Gamma}_0 = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 720 & & \\ & & & & 1680\tau/\sigma & \\ & & & & & 8400 \end{bmatrix} \tag{48}$$

$$\mathbf{\Phi}_0 = \begin{bmatrix} 1 & -\alpha & 0 & \sqrt{5} & 0 & \sqrt{7} \\ 0 & 2\alpha & 0 & -6\sqrt{5} & 0 & -12\sqrt{7} \\ 0 & 0 & 2\epsilon & 0 & -2\alpha\epsilon & 0 \\ 1 & \alpha & 0 & \sqrt{5} & 0 & -\sqrt{7} \\ 0 & 2\alpha & 0 & 6\sqrt{5} & 0 & -12\sqrt{7} \\ 0 & 0 & 2\epsilon & 0 & 2\alpha\epsilon & 0 \end{bmatrix} \tag{49}$$

where $\epsilon = \sqrt{35/\sigma}$. The first two modes are rigid body bending modes, the third the rigid body twist. (We started from the consistent mass matrix, but we could equally well have begun with the lumped mass matrix.)

We now consider what families we may generate from $\mathbf{K}_0$, $\mathbf{M}_0$. There are 6 dof, so that the generic family would involve the three 3×3 matrices $\mathbf{S}_R$, $\mathbf{S}_{RS}$, $\mathbf{S}_S$ and the three non-zero eigenvalues in $\mathbf{\Gamma}$. Such matrices might be needed in modeling an element adjacent to a joint, where no assumptions could be made regarding symmetry. This is the most general situation. Let us consider a more special situation in detail. Suppose that the element has a defect which causes the centre of shear to shift from the centre of mass of the cross section and so introduces coupling between bending and twist, as shown in Fig. 2. Let us suppose that this defect does no affect the lengthwise symmetry of the element, nor its mass and moments of inertia. What family of matrices would be needed to model this situation?

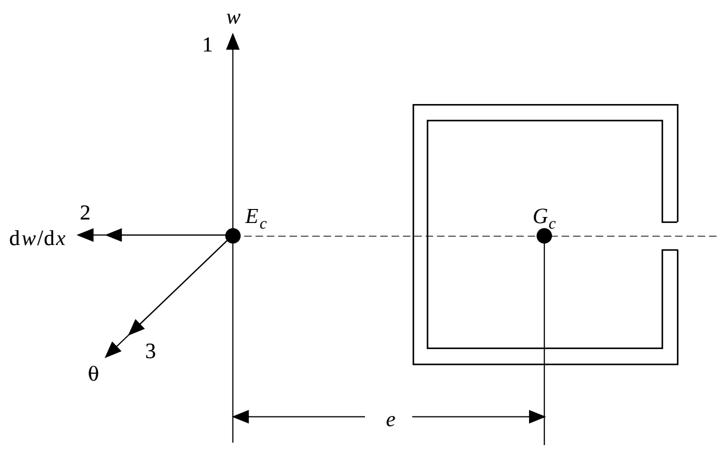


Figure 2. Defect causing the shear centre to shift from the mass centre.

First consider the rigid body modes. The first two, the bending modes will be unchanged, just the third will be changed; it will be a combination of the twisting mode and the two bending modes, so that the relationship will take the form

$$\mathbf{\Phi}_{0R}=\mathbf{\Phi}_R\mathbf{S}_R, \quad \text{where } \mathbf{S}_R=\begin{bmatrix} 1 & 0 & S_{13} \\ 0 & 1 & S_{23} \\ 0 & 0 & S_{33} \end{bmatrix}. \tag{50}$$

If the element is to retain its axial moment of inertia, then

$$S_{13}^2+S_{23}^2+S_{33}^2=1.$$

We could argue further that the coupling between the second, antisymmetrical, bending mode, and the twist, will be less than that between the first and the twist, so that S_{23} could be neglected compared to S_{13}.

We can argue the same way regarding $\mathbf{S}_{RS}$ and $\mathbf{S}_S$: that the dominant coupling will be between the fourth, symmetrical bending, mode and the rigid body twisting, mode three; and between the fifth, antisymmetrical twisting, mode, and second, antisymmetrical, rigid body mode. This would mean

$$\mathbf{S}_{RS}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{25} & 0 \\ S_{34} & 0 & 0 \end{bmatrix}, \quad \mathbf{S}_S=\begin{bmatrix} S_{44} & 0 & 0 \\ 0 & S_{55} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{51}$$

The family thus formed includes the mass matrix derived by Hallauer and Liu [13], which incorporates just the S_{13} and S_{25} coupling, with

$$S_{13}=S_{25}=-\frac{ye}{21}, \quad S_{34}=0, \quad S_{44}=1, \quad S_{25}^2+S_{55}^2=1 \tag{52}$$

where $y=42e/L$ and e is the distance between the two centres. In parameter updating the family specified by equation (51), with, say, $S_{44}=1$, $S_{25}^2+S_{55}^2=1$, would provide a large enough family to accommodate symmetrical elements with possible defects.

6. AN ALTERNATIVE WAY TO SET UP FAMILIES

Earlier, we started with the eigenvalue problem for the element; we coupled **K** and **M**. Instead, we may consider **K** and **M** separately, writing

$$\mathbf{K}=\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T=\sum_{i=d+1}^{r}\lambda_i\mathbf{u}_i\mathbf{u}_i^T, \tag{53}$$

$$\mathbf{M}=\mathbf{V}\boldsymbol{\Sigma}\mathbf{V}^T=\sum_{i=1}^{r}\sigma_i\mathbf{v}_i\mathbf{v}_i^T, \tag{54}$$

where, **U**, **V** are orthogonal matrices and

$$\mathbf{U}^T\boldsymbol{\Phi}_R=0. \tag{55}$$

We may thus define a family by starting from some original model $\mathbf{K}_0$ and $\mathbf{M}_0$ and defining the new **U** and **V** matrices by

$$\mathbf{U}=\mathbf{U}_0\mathbf{R},\quad \mathbf{V}=\mathbf{V}_0\mathbf{T}, \tag{56}$$

where **R**, **T** are orthogonal matrices, and, if need be, by specifying new values of $(\lambda_i)_{d+1}^r$, $(\sigma_i)_1^r$. Thus

$$\mathbf{K}=\mathbf{U}_0\mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^T\mathbf{U}_0^T, \tag{57}$$

$$\mathbf{M}=\mathbf{V}_0\mathbf{T}\boldsymbol{\Sigma}\mathbf{T}^T\mathbf{V}_0^T. \tag{58}$$

Again, when the element belongs to some symmetry group, the eigenvectors $\mathbf{u}_i$, $\mathbf{v}_i$ reflect the symmetry properties of the group, and **R**, **T** are made up of diagonal blocks.

To reduce the number of unknowns in the symmetric products $\mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^T$ and $\mathbf{T}\boldsymbol{\Sigma}\mathbf{T}^T$, we can update only the dominant modes of each matrix and keep the remainder unchanged. Ross [14] has suggested that in modeling a dynamic system, with **K** and **M**, it is important to ensure that **M** and the flexibility matrix **F** are modeled correctly. This means that we must correctly model the higher modes of **M** and the lower modes of **K**.

We illustrate this formulation by considering families of plate bending elements. Here however we encounter a bewildering array of elements: triangular, rectangular, quadrilateral and isoparametric; conforming and non-conforming; Kirchhoff, Reissner, Mindlin or hybrid; 3 or more d.o.f. per node; etc. In updating, the question of whether the element is conforming or not is of secondary importance. Our primary aim is to find a family of elements amongst which we can search for the most appropriate. For brevity we consider just one family, of 9 dof triangular elements, with one axis of symmetry, as shown in Fig. 3. This element has 9 dof; there are three rigid-body modes and six strain modes; for the latter there are three modes symmetrical about the axes of symmetry which we label u_1, u_3, u_5, three which are anti-symmetrical, which we label u_2, u_4, u_6. Suppose we start with the so-called BCIZ1 element derived by Bazeley, Cheung, Irons and Zienkiewicz [15] one that is simple to construct, but non-conforming. We could form a family by taking

$$\mathbf{U}=[u_1, u_3, u_5, u_2, u_4, u_6]\begin{bmatrix}\mathbf{R}_S & \\ & \mathbf{R}_A\end{bmatrix} \tag{59}$$

where $\mathbf{R}_S$ and $\mathbf{R}_A$ are two 3×3 orthogonal matrices. The family would have 12 parameters: three for each of $\mathbf{R}_S$, $\mathbf{R}_A$, and six eigenvalues $(\lambda_i)_4^9$. One of the elements in this family is the so-called DKT element. See Batoz, Bathe and Ho [16] for a discussion of this and other

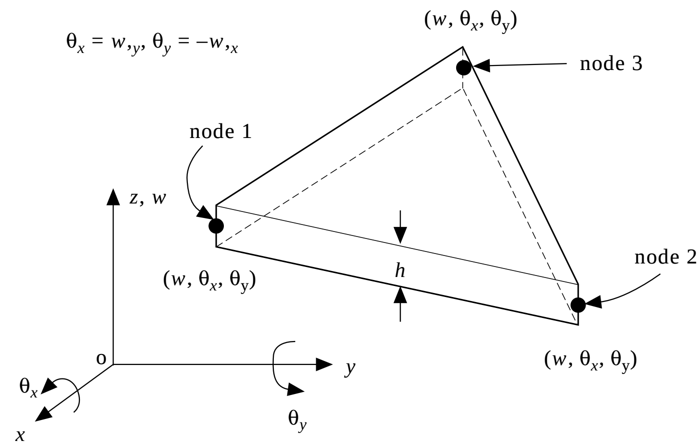


Figure 3. A 3-noded, 9 dof triangular plate element.

triangular elements. A simple calculation shows that when $v = 1/3$ the appropriate matrices $\mathbf{R}_S$ and $\mathbf{R}_A$ are

$$\mathbf{R}_S = \begin{bmatrix} -0.8606 & 0.5090 & 0.0196 \\ -0.5080 & -0.8549 & -0.1050 \\ -0.0367 & -0.1003 & 0.9943 \end{bmatrix},$$

$$\mathbf{R}_A = \begin{bmatrix} 0.9985 & -0.0297 & 0.0458 \\ -0.0309 & 0.9992 & -0.0254 \\ 0.0450 & 0.0268 & 0.9986 \end{bmatrix}. \tag{60}$$

Remembering that the three lowest modes have the most effect in the vibration analysis, we may choose to keep the highest three modes u_4, u_5, u_6 unchanged. But if we keep two of the antisymmetrical modes unchanged, then the third, u_2, will be unchanged too, so that $\mathbf{R}_A = \mathbf{I}$. Thus the updating is confined to $\mathbf{R}_S$ which, since u_5 is unchanged, will have the form

$$\mathbf{R}_S = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{61}$$

By comparing these with the matrices given in (60) we see that even with the family specified by the three parameters λ_1, λ_3, α we can get very near the DKT model, and so have quite a large family of models.

7. CONCLUSIONS

The process of finite element model updating requires that we have families of mass and stiffness matrices, assembled from element matrices, amongst which we may search for the most appropriate one. We have shown two different ways in which such families of element matrices may be constructed. Our paper has been concerned only with this first phase of model updating: reduction of the problem to a parameter identification problem. The second phase: identification, or optimal choice of the parameters to fit behavioral data, constitutes a separate problem which is discussed in Ahmadian, Gladwell and Ismail [16].

There is a third phase in the updating: interpretation of the results. Our discussion and examples of generic elements has shown exactly how many stiffness and mass parameters there can be in any parameter family of elements. Assigning intuitive meanings to these various parameters may sometimes be difficult, but at least it is a more meaningful problem than that of assigning intuitive meaning to the entries in matrices which have been derived solely on the basis of some best fit criterion, without insisting that they be derivable from assembling some generic element matrices.

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