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Short Communication

The Courant-Herrmann conjecture

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The Courant-Herrmann Conjecture (CHC) concerns the sign properties of *combinations* of the Dirichlet eigenfunctions of elliptic pde's, the most important of which is the Helmholtz equation $\Delta u + \lambda \rho u = 0$ for $D \in \mathbb{R}^N$. If the eigenvalues are ordered increasingly, CHC states that the nodal set of a combination $v = \sum_{i=1}^{n} c_i u_i$ of the first n eigenfunctions, divides D into *no more than n sign domains* in which v has one sign. The conjecture is classically known to hold for $N = 1$, we conjecture that it is true for rectangular boxes in $\mathbb{R}^N (N \geq 2)$, but show that it is false in general.

1 Introduction

The time-reduced infinitesimal vibration of a one-dimensional system, such as a rod in longitudinal or torsional vibration, or a taut string in transverse vibration, is governed by an ordinary differential equation of Sturm-Liouville type. Thus for a taut string under tension T, with density $\rho(x)$, vibrating with frequency ω, the equation governing the transverse deflection $u(x,t) = u(x) \sin \omega t$, takes the form

$$u'' + \lambda \rho u = 0, \qquad \lambda = \omega^2/T\,. \tag{1}$$

For a string with fixed ends 0, 1, the end conditions are of Dirichlet type: $u(0) = 0 = u(1)$. The solutions of eq. (1), under Dirichlet end conditions, have three simple properties (Gantmakher and Krein [6] and Gladwell [7]):

i) the eigenvalues λ are simple, i.e.,

$$0 < \lambda_1 < \lambda_2 < \dots\,.$$

ii) the nth eigenfunction u_n has exactly $n-1$ simple nodes in $(0,1)$. At each simple node, u_n changes sign, so that u_n, by its simple nodes, divides $(0,1)$ into *exactly n sign-domains* in which u_n has one sign.

In order to state the third property we must distinguish two kinds of zeros of a continuous function: a simple node, where it changes sign; a *null anti-node* where it retains its sign. The third property is

iii)

$$u = \sum_{i=p}^{q} c_i u_i, \qquad 1 \leq p \leq q\,,$$

has no less than $p-1$ simple nodes and no more than $q-1$ zeros in $(0,1)$; in this count each null anti-node counts as two zeros.

In particular therefore if u has $q-1$ different zeros in $(0,1)$, then all these zeros are simple nodes. Taking $p = 1, q = n$ we may state that the zeros of a combination of the first n modes $(u_i)_1^n$ divides $(0,1)$ into no more than n sign domains. Courant's Nodal Line Theorem (CNLT) (Courant and Hilbert [3], and Pockels [10]) concerns the generalization of property ii) to the Dirichlet eigenfunctions of elliptic equations, the simplest and most important of which is the Helmholtz equation

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$$\Delta u + \lambda \rho u = 0, \qquad u \in D\,. \tag{2}$$

Here Δ is the Laplacian, ρ is bounded and positive, and D is a domain in $\mathbb{R}^N$.

The nodal set of u is defined as the set of points where u vanishes. It is known (Cheng [2]) that for $D \subset \mathbb{R}^N$, the nodal set of an eigenfunction of (2) consists of $(N-1)$-dimensional hypersurfaces. These hypersurfaces cannot end in the interior of D, which implies that they are either closed, or begin and end at the boundary. In particular therefore in $\mathbb{R}^2$, the nodal set of an eigenfunction u of (2) is made up of nodal curves, which are either closed, or begin and end at the boundary. CNLT states that if the eigenvalues $(\lambda_i)_1^n$ are ordered increasingly, and for $N \geq 2$ they are not necessarily distinct, then each eigenfunction u in the eigenspace of λ_n divides D, by its nodal set, into *at most* n subdomains, called *nodal domains*, or the more informative *sign domains*, in which u has one sign. This is the generalization of ii).

In a footnote on p. 454 of Courant and Hilbert [3] it is stated that H. Herrmann, in his 1932 Göttingen dissertation [9], proved a generalization of iii) which we call the Courant-Herrmann Conjecture (CHC): any combination $v = \sum_{i=1}^n c_i u_i$ of the first n eigenfunctions of (2) divides D by its nodal set into at most n sign domains. We examined Herrmann's dissertation and his subsequent publications and found that he had not even stated, let alone proved, this result. Because the nodal set of a combination of eigenfunctions can exhibit far greater variety than that of a single eigenfunction, it is clear that in the statement that the nodal set of v 'divides D into sign domains' the term 'sign domain' must be interpreted in a weak sense, ≥ 0 or ≤ 0. Thus it is shown in §3 that in $\mathbb{R}^2$ the nodal set of a combination may contain isolated nodal *points*, in addition to nodal curves; in $\mathbb{R}^3$ the nodal set may contain isolated nodal points and curves in addition to nodal surfaces.

Nevertheless, the CHC does appear to hold for at least some domains $D \subset \mathbb{R}^N$ for $N \geq 2$, in particular for square domains in $\mathbb{R}^2$, as we show in §2. However, as we prove in §3 it is false in general. We find a domain $D \subset \mathbb{R}^2$ for which $v = c_1u_1 + c_2u_2$, computed by MATLAB, has 3 sign domains. Not only is CHC false, but we conjecture that given m, we may find $D \subset \mathbb{R}^2$, such that a combination, say $c_1u_1 + c_2u_2$, may have more than m sign domains; we exhibit domains D for which this combination has 3, 4, and 5 sign domains. Note that Arnol'd [1], working in an extremely abstract context, proved that (the analogue of) CHC, in his context, was false; he did not present a counterexample. We conclude that CHC is true for combinations of the first 13 modes of a square membrane. We conjecture that CHC is true for any combinations of modes of a square, rectangle, or other convex membrane.

There is a discrete analogue of eq. (1) and corresponding discrete analogues of properties i)-iii). The discrete analogue of eq. (1) is the matrix equation

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u} = 0, \tag{3}$$

where $\mathbf{A} = \mathbf{A}(n \times n)$ is a symmetric tridiagonal matrix with strictly negative codiagonal. It is known (Gantmakher and Krein [6] and Gladwell [7]) that

i) the eigenvalues of (3) are simple, i.e.,

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n\,.$$

To state the discrete form of ii) one must introduce the sign counts $S_{\mathbf{u}}^+, S_{\mathbf{u}}^-$; these, the discrete replacements for the concepts of simple node and null anti-node, are the maximum and minimum numbers of sign changes in the sequence $u_1, u_2, \ldots, u_n$ obtained by assigning $+$ or $-$ to any zero terms. The discrete form of ii) is

ii′) if $\mathbf{u}_i$ is the eigenvector corresponding to λ_i, then

$$S_{\mathbf{u}}^+ = S_{\mathbf{u}}^- = i - 1\,.$$

This implies that the first and last components $u_{1,i}, u_{n,i}$ are non-zero, and if $u_{m,i} = 0$ $(1 < m < n)$ then $u_{m-1,i}, u_{m+1,i}$ are non-zero and have opposite signs.
The discrete analogue of iii) is

iii′)

$$\mathbf{u} = \sum_{i=p}^{q} c_i \mathbf{u}_i, \quad 1 \leq p \leq q \leq n, \quad \sum_{i=p}^{q} c_i^2 > 0\,,$$

then

$$p - 1 \leq S_{\mathbf{u}}^- \leq S_{\mathbf{u}}^+ \leq q - 1\,.$$

We conclude that there is a clear correspondence between (1) and (3), and between i)-iii) and i′)-iii′).

Recent research has shed light on discrete counterparts of these results for matrices other than tridiagonal. These are conveniently described by using graph theory. Suppose $\mathcal{G}$ is a simple undirected graph without loops on N vertices $(P_i)_1^N$

making up the vertex set $\mathcal{V}$. Let $\mathbf{A}$ be a symmetric matrix of order N with the property that if $i \neq j$ then

$$\begin{aligned} a_{ij} &< 0 \quad \text{if } (P_i, P_j) \in \mathcal{V}, \\ a_{ij} &= 0 \quad \text{otherwise}. \end{aligned}$$

If $\mathbf{u} = \{u_1, u_2, ..., u_N\}$, we now assign a sign $+, 0, -$ to P_i according to whether u_i is positive, zero, or negative. Now we introduce strong (weak) sign graphs as maximal connected subgraphs of $\mathcal{G}$ on which the vertices have the same strict (loose) signs. The discrete counterpart of ii) is then given by Davies, Gladwell, Leydold, and Stadler [4] as ii$''$). If $\mathbf{u}$ is an eigenvector corresponding to λ_n, and λ_n has multiplicity r, then $\mathbf{u}$ has at most $n + r - 1$ strong sign graphs, and at most n weak sign graphs. Gladwell and Zhu [8] showed that when λ_n is an eigenvalue of multiplicity r the eigenspace of λ_n may be spanned by r orthonormal vectors $\mathbf{u}_j,\ j = n, n+1, \ldots, n+r-1$, such that $\mathbf{u}_j$ has at most j strong sign graphs. Alternatively, the eigenspace of λ_n may be spanned by r linearly independent vectors $\mathbf{u}_j,\ j = n, n+1, ..., n+r-1$, such that each $\mathbf{u}_j$ has at most n strong sign graphs. In §3 we show that there is no simple analogue of iii) for eigenvectors on a graph. References relating to the history of these problems may be found in Davies, Gladwell, Leydold, and Stadler [4] and Gladwell and Zhu [8].

2 CHC for rectangles

For the square $D = (0,1) \times (0,1)$ the eigenvalues and eigenfunctions of (2) for $\rho \equiv 1$ are

$$\lambda_p = \lambda_{m,n} = \pi^2(m^2 + n^2), \quad u_p = u_{m,n} = \sin m\pi x \sin n\pi y\,.$$

Since we are concerned only with sign domains, we may scale x and y and consider

$$u_{m,n} = \sin mx \sin ny \qquad \text{in} \quad D' \equiv (0,\pi) \times (0,\pi). \tag{4}$$

First we show that the nodal set of a combination may consist of a single isolated point:

$$u = 2u_{1,1} + u_{1,3} + u_{3,1} = 4 \sin x \sin y (\cos^2 x + \cos^2 y)$$

has an isolated nodal point $(\pi/2, \pi/2)$. This single example is sufficient to show that sign domains must be interpreted in a weak sense. Under this interpretation, this combination u has just one sign domain: it satisfies $u \geq 0$ throughout D.

Now we use the result

$$\sin mx = \sin x U_{m-1}(\cos x)\,,$$

where U_{m-1} is a Chebyshev polynomial of the second kind. Since $\sin x \sin y$ is positive in D', the sign properties of $u_{m,n}$ are the same as those of

$$v_{m,n} = U_{m-1}(\cos x) U_{n-1}(\cos y)\,.$$

Furthermore, since the correspondence $\cos x \to X$; $(0,\pi) \to (-1,1)$ is $1:1$, we may equivalently study the properties of

$$w_{m,n} = U_{m-1}(X) U_{n-1}(Y), \qquad (X,Y) \in (-1,1) \times (-1,1)\,. \tag{5}$$

In the absence of an analytical proof, we had to resort to an examination of individual cases, a tedious process. The order in which the eigenvalues λ_p appear is shown in the first column of Table 1. The second column shows the values of m, n. The third column shows the new high order terms which appear in combination of the first p eigenfunctions. The last column shows the nodal curves of *a* combination which exhibits the maximum member of sign domains. These combinations were found empirically.

This empirical examination shows that, for the cases considered, CHC does hold. We examined the conjecture for rectangles also. Now

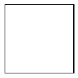
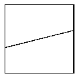
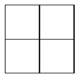
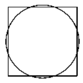
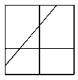
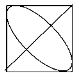
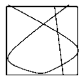
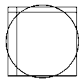
$$\lambda_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \qquad u_{m,n} = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

Again we may scale x and y, and consider $u_{m,n}$ given by (5); the difference is merely that the order in which the $\lambda_{m,n}$ appear depends on the ratio $a : b$. We were unable to find a counterexample to CHC.

As we have shown, the problem reduces to the question of how many regions a polynomial $P_n(x,y)$ divides a square, by its nodal places. This problem is still open, as is the problem of how many regions a polynormal divides the whole plane, by its nodal places.

It is instructive to consider how the original result iii) is proved when $N = 1$. It is proved ([6], [7]) by replacing the differential eigenvalue problem by an integral one, by considering compound operators on a higher dimensional product space, and noting that certain determinants formed from the original eigenfunctions are fundamental Dirichlet eigenfunctions on one orthant $0 \leq x_1 \leq x_2 \leq \ldots \leq x_p \leq 1$ of the product space. It is not possible to carry this argument over to the case in which $D = (0,1) \times (0,1)$.

Table 1 CHC is true for the first 13 eigenfunctions on the square.

	m, n	New High Order Terms		Maximum No. of Nodal domains	
1	1, 1	1		1	
2, 3	1, 2 2, 1	X	Y	2	
4	2, 2	XY		4	
5, 6	1, 3 3, 1	X^2	Y^2	5	
7, 8	2, 3 3, 2	X^2Y	XY^2	7	
9, 10	1, 4 4, 1	X^3	Y^3	8	
11	3, 3	X^2Y^2		10	
12, 13	2, 4 4, 2	X^3Y	$X\ Y^3$	12	

3 CHC is false

We exhibit Fig. 1. This shows the nodal curves of a combination $v = u_2 + cu_1$ for three values of c, as calculated by MATLAB. When $c = 0$, $v = u_2$ has just one nodal curve, two sign domains. As c increases, the nodal curve rises and eventually (for $c \approx 0.96$) makes a discontinuous jump into two separate curves, producing two disjoint negative sign domains, and thus three sign domains in all. This single counterexample is sufficient to disprove CHC for general domains. Admittedly the modes have been calculated by MATLAB and not analytically, but in such a simple situation there is little reason to doubt the accuracy of the MATLAB solution. Fig. 2 shows two other examples of domains in which v has 3 or 4 negative sign domains, and thus 4 and 5 sign domains in all. These examples indicate that not only is CHC false for non-convex domains, but that it is 'entirely' false and misleading in the sense that, given $m > 2$, it seems to be possible to construct a domain such that a combination of the first two modes of D has more that m sign domains.

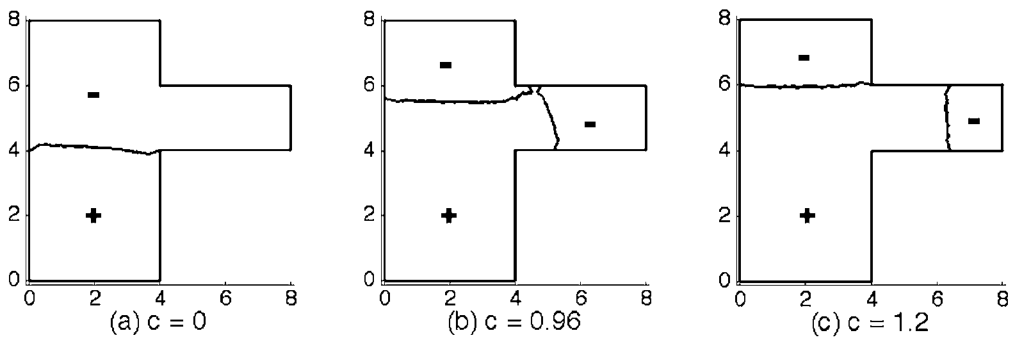


Fig. 1 The nodal lines of the linear combinations $cu_1 + u_2$ for $c = 0, 0.96, 1.2$.

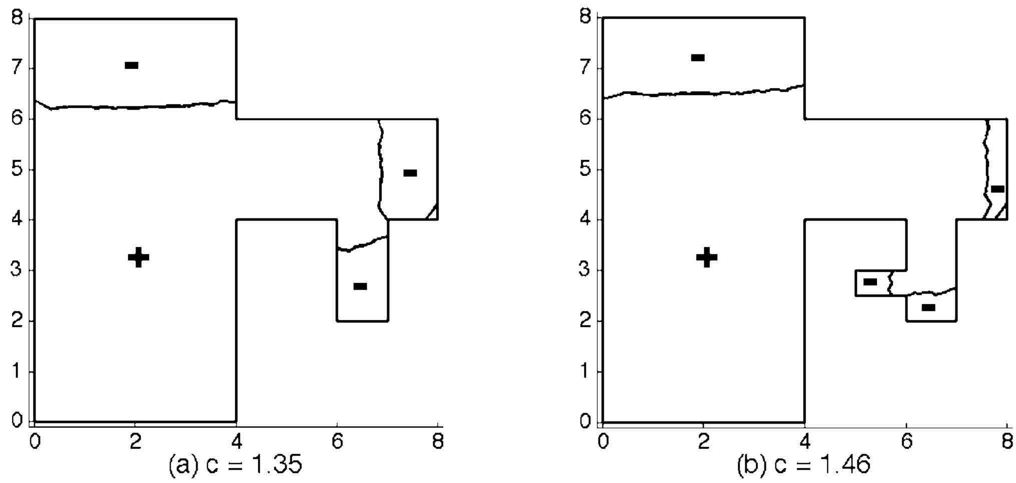


Fig. 2 The nodal lines of the linear combination $cu_1 + u_2$ may divide the domain into a) four or b) five sign domains.

To show that there is no discrete counterpart of CHC, i.e., of iii) for a general graph, or even a tree, we consider the star on N points shown in Fig. 3, and consider the matrix

$$A = \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ -1 & 2 & & & \\ -1 & & 3 & & \\ \vdots & & & \ddots & \\ -1 & & & & N \end{pmatrix} .$$

The eigenvalues are the roots of

$$f(\lambda) \equiv 1 - \lambda - \sum_{i=2}^{N} \frac{1}{i - \lambda}$$

so that

$$\lambda_1 < 2 < \lambda_2 < \dots < \lambda_{N-1} < N < \lambda_N$$

and the eigenvectors are

$$\mathbf{x}_i = \left\{ 1, \frac{1}{2 - \lambda_i}, \frac{1}{3 - \lambda_i}, \dots, \frac{1}{N - \lambda_i} \right\} .$$

Consider combinations of $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{x} = \left\{ 1 + c, \frac{1}{2 - \lambda_1} + \frac{c}{2 - \lambda_2}, \frac{1}{3 - \lambda_1} + \frac{c}{3 - \lambda_2}, \dots, \frac{1}{N - \lambda_1} + \frac{c}{N - \lambda_2} \right\} .$$

Choose c so that $-1 < c < -(N - \lambda_2)/(N - \lambda_1)$ then $\mathbf{x}$ has signs

$$\{+, +, -, -, \dots, -\} .$$

There are $N - 1$ sign graphs: the number of sign graphs can be made as large as desired simply by increasing N.

4 A restricted theorem

In all the examples shown in §3, u has just one positive sign domain. These are examples of

Theorem 1. *Suppose D is a connected domain in $\mathbb{R}^N$, and that eq.* (2) *has eigenvalues λ_i and eigenfunctions u_i. If the eigenvalues are ordered increasingly, $n \geq 2$, $c > 0$, and $u_1 > 0$ in D, then $v = u_n + cu_1$ has at most $n - 1$* ***positive*** *sign domains.*

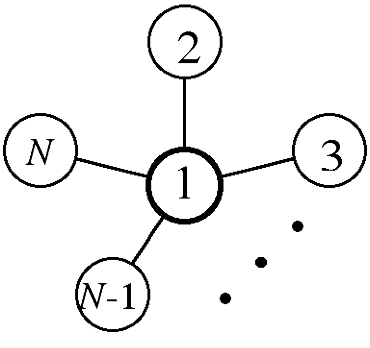


Fig. 3 A star on N points.

Proof. Define

$$[u, v]_D = \int \rho u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}\,, \quad (u, v)_D = \int \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}\,.$$

Suppose v has m *positive* sign domains $(D_i)_1^m$. Define

$$w_i(\mathbf{x}) = \begin{cases} \beta_i v(\mathbf{x})\,, & \mathbf{x} \in D_i\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Since the D_i are disjoint, $(w_i)_1^m$ are orthogonal. Scale the w_i, i.e., choose the $\beta_i > 0$, so that $[w_i, w_i]_{D_i} = 1$. Now compute $(w_i, w_i)_{D_i}$. We have

$$\begin{aligned} (w_i, w_i)_{D_i} &= \int_{D_i} \nabla w_i \cdot \nabla w_i d\mathbf{x} \\ &= \int_{D_i} \{\div(w_i \nabla w_i) - w_i \Delta w_i d\mathbf{x} \\ &= \int_{D_i} w_i \frac{\partial w_i}{\partial n} ds + \int_{D_i} \rho w_i(\lambda_n u_n + \lambda_1 c u_1) d\mathbf{x} \end{aligned}$$

and since $w_i = 0$ on ∂D_i we have

$$(w_i, w_i)_{D_i} = [w_i, \lambda_n u_n + \lambda_1 c u_1]_{D_i}\,.$$

Now we take

$$u(\mathbf{x}) = \sum_{i=1}^m c_i w_i(\mathbf{x}), \qquad \sum_{i=1}^m c_i^2 = 1\,,$$

and compute the Rayleigh Quotient for $u(\mathbf{x})$:

$$[u, u]_D = \sum_{i=1}^m c_i^2 [w_i, w_i]_{D_i} = \sum_{i=1}^m c_i^2 = 1\,;$$

$$(u, u)_D = \sum_{i=1}^m c_i^2 (w_i, w_i)_{D_i} = \sum_{i=1}^m c_i^2 [w_i, \lambda_n u_n + \lambda_1 c u_1]_{D_i}\,.$$

But on D_i, $\lambda_n u_n + \lambda_1 c u_1 = \lambda_n(u_n + c u_1) + (\lambda_1 - \lambda_n) c u_1 = \lambda_n w_i + (\lambda_1 - \lambda_n) c u_1$, so that

$$(u, u)_D = \lambda_n \sum_{i=1}^m c_i^2 [w_i, w_i] + (\lambda_1 - \lambda_n) \sum_{i=1}^m c_i^2 [w_i, c u_1]_{D_i}\,.$$

The *Rayleigh Quotient* for u is

$$\lambda_R \equiv \frac{(u, u)_D}{[u, u]_D} = \lambda_n - (\lambda_n - \lambda_1) c \sum_{i=1}^m c_i^2 [w_i, u_1]_{D_i}\,. \tag{6}$$

Now choose $(c_i)_1^m$ so that $[u, u_j] = 0$, for $j = 1, 2, \ldots, m-1$, then the minmax theorem for the Rayleigh Quotient states that $\lambda_R \geqslant \lambda_m$, with equality if and only if $u = u_m$. On the other hand, since D is *connected*, λ_1 is simple (Pockels [10]) so that $\lambda_n - \lambda_1 > 0$; since the D_i are, by hypothesis, *positive* sign domains, we have $[w_i, u_1]_{D_i} > 0$; hence eq. (6) states that $\lambda_R < \lambda_n$ so that $\lambda_m < \lambda_n$ and $m < n$, i. e., $m \leqslant n - 1$.

We note that Fiedler [5] proved a discrete analogue of this result for eigenvectors on a graph.

5 Conclusions

The Courant Herrmann Conjecture appears to be true for a restricted class of domains, but whether it is true even for squares in $\mathbb{R}^2$ is still open. It is false in general.

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