Inverse Problems 7 (1991) 557–565. Printed in the UK

# The application of Schur's algorithm to an inverse eigenvalue problem

G M L Gladwell

Solid Mechanics Division, Faculty of Engineering, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

Received 30 October 1990

**Abstract.** Andersson recently developed an algorithm for the inverse eigenvalue problem for the Sturm–Liouville equation in impedance form. The impedance $A(x)$ is assumed to be piecewise constant over $N$ equal intervals. In this paper we show that Andersson's algorithm is equivalent to Schur's algorithm, which is well known in one-dimensional seismology and transmission line theory, and is known to be numerically efficient and stable.

## 1. Introduction

In a recent paper, Andersson [1] considered the inverse eigenvalue problem for the Sturm–Liouville equation in impedance form, namely

$$(A(x)w'(x))' + \omega^2 A(x)w(x) = 0 \qquad 0 \leqslant x \leqslant L \tag{1.1}$$

subject to the end conditions

I. $\qquad w'(0) = 0 = w'(L)$

II. $\qquad w(0) = 0 = w'(L)$.

Equation (1.1) describes the infinitesimal, free, longitudinal vibrations of a thin, straight rod of cross-sectional area $A(x)$ under 'free–free' (I) or 'fixed–free' (II) end conditions; $\omega$ is the (scaled) natural frequency. He showed that if there were given eigenvalues $(\omega_k)_0^N$ such that

$$0 = \omega_0 < \omega_1 < \cdots < \omega_N = \pi N/2L \tag{1.2}$$

and such that the even $\omega_j$ are eigenvalues for I, the odd for II, then there exists a unique rod with piecewise constant $A(x)$, such that

$$A(x) = A_j \qquad (j-1)\Delta < x \leqslant j\Delta \tag{1.3}$$

where $\Delta = L/N$, $j = 1, 2, \ldots, N$, $A_1 = 1$. He presented an algorithm for the determination of the $A_j$.

0266-5611/91/040557+09$03.50  

The purpose of this paper is to place Andersson's algorithm within the context of inversion algorithms in seismology and transmission line theory [2, 3]. In that context, a medium with parameters that are piecewise constant over equal intervals of depth $\Delta$, such as (1.3), is called a *Goupillaud medium*. In seismology and transmission line theory, as in most inverse scattering problems, the data does not relate to eigenvalues; there are no eigenvalues, or so-called bound states. Instead the data refers to the response to an input. One way of expressing the data uses the reflected wave $U(t)$ at equal intervals $2\Delta$, due to an incoming wave $D(t)$ also sampled at intervals $2\Delta$. One of the fundamental questions is to ask whether a given reflected wave and incoming wave actually correspond to a Goupillaud medium. This is the question: 'Is the data realizable?' The realizability criterion can be phrased by introducing the $Z$-transforms, $U(z)$ and $D(z)$, of $U(t)$ and $D(t)$, and defining the left-reflection function

$$R(z) = \frac{U(z)}{D(z)} \tag{1.4}$$

and then putting

$$f_1(z) = z^{-1}R(z). \tag{1.5}$$

The criterion is

$$M(f_1) \equiv \sup_{|z|\leqslant 1} | f_1(z) | \leqslant 1. \tag{1.6}$$

Schur [4] constructed an algorithm to test whether a function $f_1(z)$ satisfies (1.6), that is, is bounded by 1 on the unit disc. It is based on the fact that if $|\gamma| < 1$, then

$$w = \frac{z-\gamma}{1-\bar{\gamma}z} \tag{1.7}$$

maps $|z| \leqslant 1$ onto $|w| \leqslant 1$ and $|z| = 1$ onto $|w| = 1$. Schur's algorithm is based on the recurrence

$$f_j(z) = \frac{1}{z}\frac{f_{j-1}(z)-\gamma_j}{1-\bar{\gamma}_j f_{j-1}(z)} \qquad j = 2, 3, \ldots \tag{1.8}$$

where $\gamma_j = f_{j-1}(0)$. Suppose $M(f_{j-1}) \leqslant 1$. There are two possibilities: either $|\gamma_j| = 1$, in which case the condition $M(f_{j-1}) \leqslant 1$ and the maximum modulus principle (Knopp [5], p 84) forces $f_{j-1}(z) = \gamma_j$, so that the sequence terminates at $f_{j-1}(z)$; or $|\gamma_j| < 1$, in which case $M(f_j) \leqslant 1$. Thus the condition (1.6) used with the recurrence (1.8) leads to a finite or infinite sequence, $\gamma_2, \gamma_3, \ldots$, with the property $|\gamma_j| \leqslant 1$, where the inequality is strict except possibly for the last one. We note in particular that if $M(f_1) = 1$, then $M(f_j) = 1$ for all $j$, and if the sequence terminates at $j = N+1$, it will do so with $|f_N| = 1$.

## 2. Formulation

First we replace equation (1.1) by two coupled first-order equations, namely

$$w'(x) = \mathrm{i}\omega p(x)/A(x) \tag{2.1}$$

$$p'(x) = \mathrm{i}\omega A(x)w(x). \tag{2.2}$$

Put

$$\eta(x) = \{A(x)\}^{1/2} \tag{2.3}$$

and define down and up quantities

$$D = \tfrac{1}{2}\{\eta w + \eta^{-1} p\} \tag{2.4}$$
$$U = \tfrac{1}{2}\{\eta w - \eta^{-1} p\}. \tag{2.5}$$

These satisfy the equations

$$D' = \mathrm{i}\omega D + \eta'\eta^{-1} U \tag{2.6}$$
$$U' = -\mathrm{i}\omega U + \eta'\eta^{-1} D \tag{2.7}$$

so that if $A(x) =$ constant, then $\eta' = 0$ and

$$D' = \mathrm{i}\omega D \qquad U' = -\mathrm{i}\omega U \tag{2.8}$$

which have the solutions

$$D = D_0 \mathrm{e}^{\mathrm{i}\omega x} \qquad U = U_0 \mathrm{e}^{-\mathrm{i}\omega x}. \tag{2.9}$$

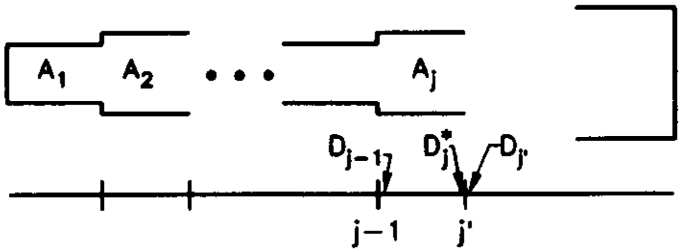


**Figure 1.** Beam of piecewise constant cross-section.

Suppose $A(x)$ has the form (1.3). Define the quantities

$$D_j = D(j\Delta+) \qquad U_j = U(j\Delta+) \qquad D_j^* = D(j\Delta-) \qquad U_j^* = U(j\Delta-). \tag{2.10}$$

Then equations (2.9) and figure 1 show that

$$D_j^* = \mathrm{e}^{\mathrm{i}\omega\Delta} D_{j-1} \qquad U_j^* = \mathrm{e}^{-\mathrm{i}\omega\Delta} U_{j-1}. \tag{2.11}$$

Put $\mathrm{e}^{\mathrm{i}\omega\Delta} = z^{1/2}$, then

$$\begin{bmatrix} D_j^* \\ U_j^* \end{bmatrix} = \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix} \begin{bmatrix} D_{j-1} \\ U_{j-1} \end{bmatrix}. \tag{2.12}$$

Let

$$\mathsf{H}_j = \frac{1}{2} \begin{bmatrix} \eta_j & \eta_j^{-1} \\ \eta_j & -\eta_j^{-1} \end{bmatrix}. \tag{2.13}$$

Then equations (2.4), (2.5) and the continuity of $w$ and $p$ across a discontinuity of $A(x)$ give

$$\begin{bmatrix} D_{j-1} \\ U_{j-1} \end{bmatrix} = \mathbf{H}_j \begin{bmatrix} w_{j-1} \\ p_{j-1} \end{bmatrix} \qquad \begin{bmatrix} D^*_{j-1} \\ U^*_{j-1} \end{bmatrix} = \mathbf{H}_{j-1} \begin{bmatrix} w_{j-1} \\ p_{j-1} \end{bmatrix} \tag{2.14}$$

so that

$$\begin{bmatrix} D_{j-1} \\ U_{j-1} \end{bmatrix} = \mathbf{H}_j \mathbf{H}_{j-1}^{-1} \begin{bmatrix} D^*_{j-1} \\ U^*_{j-1} \end{bmatrix}. \tag{2.15}$$

The matrix

$$\Theta_j = \mathbf{H}_j \mathbf{H}_{j-1}^{-1} \tag{2.16}$$

may be written

$$\Theta_j = \frac{1}{\sigma_j} \begin{bmatrix} 1 & -\gamma_j \\ -\gamma_j & 1 \end{bmatrix} \tag{2.17}$$

where

$$\sigma_j = (1 - \gamma_j^2)^{1/2} \qquad \gamma_j = \frac{A_{j-1} - A_j}{A_{j-1} + A_j}. \tag{2.18}$$

We can combine equations (2.11) and (2.14) to obtain

$$\begin{bmatrix} D^*_j \\ U^*_j \end{bmatrix} = \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix} \Theta_j \begin{bmatrix} D^*_{j-1} \\ U^*_{j-1} \end{bmatrix}. \tag{2.19}$$

Put

$$\frac{U^*_j}{D^*_j} = f_j(z), \tag{2.20}$$

then equation (2.18) gives

$$f_j(z) = \frac{1}{z} \cdot \frac{f_{j-1}(z) - \gamma_j}{1 - \gamma_j f_{j-1}(z)} \tag{2.21}$$

which, since $\gamma_j$ is real, is precisely Schur's recurrence (1.8).

## 3. The forward problem

Suppose that we are given the parameters $(A_j)_1^N$ and we wish to find the eigenvalues corresponding to the end conditions I and II. Suppose that the rod is vibrating with frequency $\omega$ and that the end condition at $x = L$ is satisfied, then without loss of generality we can take (see (2.1))

$$w(L) = 1 \qquad w'(L) = 0 = p(L). \tag{3.1}$$

Then $D_N^* = \eta_N/2 = U_N^*$ so that

$$f_N(z) = 1. \tag{3.2}$$

The values of $w(0) = w_0$, $p(0) = p_0$ are given by

$$D_0 = \tfrac{1}{2}\{\eta_0 w_0 + \eta_0^{-1} p_0\} \tag{3.3}$$

$$U_0 = \tfrac{1}{2}\{\eta_0 w_0 - \eta_0^{-1} p_0\} \tag{3.4}$$

so that

$$\eta_0^2 \frac{w_0}{p_0} = \frac{D_0 + U_0}{D_0 - U_0} = \frac{z^{-1/2}D_1^* + z^{1/2}U_1^*}{z^{-1/2}D_1^* - z^{1/2}U_1^*}$$

$$= \frac{1 + g(z)}{1 - g(z)} \tag{3.5}$$

where

$$g(z) = z f_1(z). \tag{3.6}$$

In the forward problem, we are given $f_N(z) = 1$ and we are given the $(\gamma_j)_2^N$ with $|\gamma_j| < 1$. We may thus compute $f_{N-1}, f_{N-2}, \ldots, f_1$, using the recurrence (2.20) in the reverse form, namely

$$f_{j-1}(z) = \frac{z f_j(z) + \gamma_j}{1 + \gamma_j z f_j(z)} \qquad j = N, N-1, \ldots, 2. \tag{3.7}$$

The mapping of $z f_j(z)$ onto $f_{j-1}(z)$ has the form (1.6). Thus the region $|z f_j(z)| \leqslant 1$ is mapped onto $|f_{j-1}(z)| \leqslant 1$, and $|z f_j(z)| = 1$ is mapped onto $|f_{j-1}(z)| = 1$. But $f_N(z) = 1$ so that each $(f_j(z))_1^N$ has $|f_j(z)| = 1$ when $|z| = 1$, i.e. when $\omega$ is real. Thus the function $w = g(z)$ maps $|z| \leqslant 1$ onto $|w| \leqslant 1$, and $|z| = 1$ onto $|w| = 1$. When $g(z)$ is expressed in terms of $z$ it has the form

$$g(z) = \frac{z P_{N-1}(z)}{Q_{N-1}(z)} \tag{3.8}$$

where $P_{N-1}$ and $Q_{N-1}$ are polynomials of degree $N-1$. Thus $g(z)$ maps the circle $|z| = 1$ into itself $N$ times.

Equation (3.7) shows that *if* $f_j(z^{-1}) = 1/f_j(z)$, *then* $f_{j-1}(z^{-1}) = 1/f_{j-1}(z)$. But $f_N(z^{-1}) = 1 = 1/f_N(z)$, so that indeed

$$f_j(z^{-1}) = 1/f_j(z) \qquad j = 1, 2, \ldots, N \tag{3.9}$$

and hence

$$g(z^{-1}) = 1/g(z). \tag{3.10}$$

The mapping of $|z| = 1$ into itself caused by $g(z)$ produces two sets of $N$ points on $|z| = 1$ of significance, namely

$$\mathcal{A} = \{z \;/\; |z| = 1 \text{ and } g(z) = 1\}$$
$$\mathcal{B} = \{z \;/\; |z| = 1 \text{ and } g(z) = -1\}.$$

The points in $\mathcal{A}$ correspond to values of $z$ for which, according to (3.5), $p_0 = 0$; the $z$ values give values of $\omega$ which are eigenvalues of I. Similarly the $z$ values in $\mathcal{B}$ give $w_0 = 0$ so that $\omega$ corresponds to an eigenvalue of II. The known interlacing of these two sets of eigenvalues means that the points of $\mathcal{A}$ and $\mathcal{B}$ will interlace on the circle $|z| = 1$. Equation (3.10) shows that if $z$ is a member of either set, then $z^{-1} = \bar{z}$ is a member of the same set. Figure 2 shows the arrangement of the two sets when $N = 2$ and $N = 3$. Since $g(1) = 1$ and $g(-1) = (-1)^N$, therefore $z = 1$ is in $\mathcal{A}$, while $z = -1$ is in $\mathcal{A}$ if $N$ is even, and in $\mathcal{B}$ if $N$ is odd. It may easily be verified that there are $N + 1$ values of $z$ in $\mathcal{A} \bigcup \mathcal{B}$ which satisfy

$$0 \leqslant \arg(z) \leqslant \pi \tag{3.11}$$

and the remaining $N - 1$ values may be obtained as $z^{-1} = \bar{z}$, where $z$ satisfies (3.10). The $N + 1$ values of $z$ satisfying (3.11) yield $N + 1$ eigenvalues $\omega_j$ satisfying (1.2). Since $z = e^{2i\omega\Delta}$ is a periodic function of $\omega$ with period $\pi/\Delta = N\pi/L$, each value of $z$ gives rise to an infinite sequence of eigenvalues with equal spacing $N\pi/L$, and each $z^{-1} = e^{-2i\omega\Delta}$ gives another such sequence. Thus the system not only has the eigenvalues $\omega_0, \omega_1, \ldots, \omega_N$, but also

$$\omega_{2N+k} = N\pi/L + \omega_k \qquad \omega_{2N-k} = N\pi/L - \omega_k \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{3.12}$$

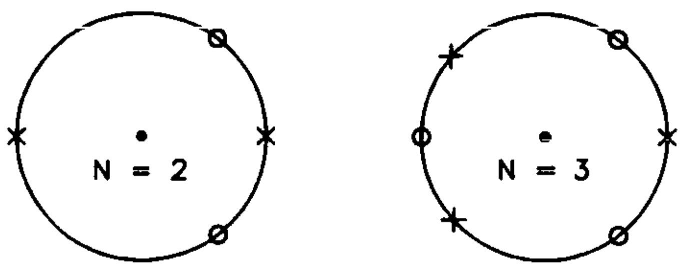


**Figure 2.** The members of $\mathcal{A}$ (×) and $\mathcal{B}$ (O) interlace on the circle.

## 4. The inverse problem

Now we are given $N+1$ eigenvalues $\omega_j$ satisfying (1.2). We must use them to construct $g(z)$ and hence $f_1(z)$, and then find the $\gamma_j$ which will lead eventually to $f_N(z) = 1$.

It is convenient to make a distinction between even and odd values of $N$. In the case when $N$ is even, let $N = 2M$. Of the $N + 1 \equiv 2M + 1$ eigenvalues, $M + 1$ are even, corresponding to I; the set $\mathcal{A}$ consists of $2M$ points: $z_0 = 1$, $z_N = -1$, and the $M - 1$ pairs $z_{2j}, z_{2j}^{-1}$, $j = 1, 2, \ldots, M - 1$. The $2M$ odd $z$'s in $\mathcal{B}$ occur in $M$ pairs $z_{2j-1}, z_{2j-1}^{-1}$, $j = 1, 2, \ldots, M$. Thus

$$\frac{\eta_0^2 w_0}{p_0} = \frac{1 + g(z)}{1 - g(z)} = \frac{C \prod_{j=1}^{M} (z - z_{2j-1})(z - z_{2j-1}^{-1})}{(z^2 - 1) \prod_{j=1}^{M-1} (z - z_{2j})(z - z_{2j}^{-1})} \tag{4.1}$$

so that $g(z) = 1$ when $z$ is a root of the denominator, and $g(z) = -1$ when $z$ is a root of the numerator. The constant, $C$ must be chosen so that $g(0) = 0$, i.e. $C = -1$; the numerator of $g(z)$ will then have no constant term in its power series expansion, while the highest powers in the denominator, $z^{2M}$, will cancel, so that $g(z)$ will have the form (3.8). Denote the right-hand side of equation (4.1) by $f(z)$, so that

$$\frac{1+g(z)}{1-g(z)} = f(z) = \zeta. \tag{4.2}$$

The function $f(z)$ will map the open, connected region

$$\mathcal{R} = \{z \;/\; |z| < 1\}$$

into an open, connected region in the $\zeta$-plane. The function $f(z)$ maps the origin $z = 0$ onto $\zeta = 1$, and $|z| = 1$ into the imaginary axis. We conclude that $f(z)$ maps $|z| \leqslant 1$ into the right-hand half-plane, so that

$$\begin{aligned} &\text{if } |z| \leqslant 1 \text{ then } \operatorname{Re} f(z) \geqslant 0 \\ &\text{if } |z| = 1 \text{ then } \operatorname{Re} f(z) = 0. \end{aligned}$$

Since the given eigenvalues, $\omega_j$, corresponding to I and II interlace, i.e. the members of $\mathcal{A}$ and $\mathcal{B}$ interlace, then as we proceed counterclockwise around $|z| = 1$ starting from $z = 1$, the points of $\mathcal{A}$ and $\mathcal{B}$ are mapped successively onto the point at infinity and the origin in the $\zeta$-plane. Equation (4.2) implies

$$g(z) = \frac{f(z)-1}{f(z)+1} = \frac{\zeta-1}{\zeta+1}. \tag{4.3}$$

But if $\operatorname{Re}\zeta \equiv \xi \geqslant 0$, then $\zeta$ is no further from 1 than it is from -1, that is $|\zeta-1| \leqslant |\zeta+1|$ so that $|g(z)| \leqslant 1$. We conclude that $g(z)$, and hence, by the Schwarz lemma, $f_1(z)$, is bounded by 1 on the unit disc.

Now apply Schur's algorithm to $f_1(z)$ to produce a sequence $\{f_j(z)\}_1^N$. The form of $g(z)$ given by (3.8) leads to a form

$$f_1(z) = P_{N-1}(z)/Q_{N-1}(z) \tag{4.4}$$

with real coefficients. Therefore all $\gamma_j$ will be real. Equation (4.1) shows that $g(z)$ has the properties

$$g(z^{-1}) = 1/g(z) \qquad g(1) = 1. \tag{4.5}$$

Therefore $f_1(z^{-1}) = 1/f_1(z)$, and $f_1(1) = 1$. Equation (1.8) now shows that

$$f_j(z^{-1}) = 1/f_j(z) \qquad f_j(1) = 1 \qquad j = 1,2,\ldots,N \tag{4.6}$$

because the statement is true for $j = 1$. Thus $f_j(z)$ will have the form

$$f_j(z) = P_{N-j}(z)/Q_{N-j}(z) \qquad j = 1,2,\ldots,N \tag{4.7}$$

so that the sequence will converge with $f_N(z) = 1$ as required, and the $\gamma_j$ will satisfy

$$-1 < \gamma_j < 1 \qquad j = 2,3,\ldots,N; \quad \gamma_{N+1} = 1. \tag{4.8}$$

Since $A_1 = 1$, by assumption, these $\gamma_j$ lead to a unique set of finite, positive $(A_j)_1^N$ as required. We stress that the single condition (1.6) ensures the existence of the $\gamma_j$ satisfying (4.8).

In the case when $N$ is odd, let $N = 2M - 1$. Of the $N + 1 \equiv 2M$ eigenvalues, $M$ are even and $M$ are odd; the set $\mathcal{A}$ consists of $z_0 = 1$ and $M - 1$ pairs $z_{2j}$, $z_{2j}^{-1}$, $j = 1, 2, \ldots, M-1$. The $2M-1$ odd $z$'s in $\mathcal{B}$ are $z_N = -1$ and $M-1$ pairs $z_{2j-1}$, $z_{2j-1}^{-1}$, $j = 1, 2, \ldots, M-1$. Now

$$\frac{\eta_0^2 w_0}{p_0} = \frac{1+g(z)}{1-g(z)} = \frac{C(z+1)\prod_{j=1}^{M-1}(z-z_{2j-1})(z-z_{2j-1}^{-1})}{(z-1)\prod_{j=1}^{M-1}(z-z_{2j})(z-z_{2j}^{-1})} \tag{4.9}$$

where again $g(0) = 0$ implies $C = -1$. Apart from this, the argument follows as before.

For computational purposes, Schur's algorithm leads to a recurrence relation for the coefficients in the polynomials $P_{N-j}(z)$ and $Q_{N-j}(z)$. Let

$$P_{N-j}(z) = \sum_{k=0}^{N-j} a_{N-j,k} z^k \qquad Q_{N-j}(z) = \sum_{k=0}^{N-j} b_{N-j,k} z^k. \tag{4.10}$$

Equation (4.10) or (4.9) yields the values of $a_{N-1,k}$ and $b_{N-1,k}$ from data. Equation (4.6) states that

$$a_{N-j,k} = b_{N-j,N-j-k} \qquad k = 0, 1, 2, \ldots, N-j. \tag{4.11}$$

The recurrence (2.20) yields

$$\gamma_j = a_{N-j+1,0}/b_{N-j+1,0} \tag{4.12}$$

$$a_{N-j,k} = a_{N-j+1,k+1} - \gamma_j b_{N-j+1,k+1} \qquad k = 0, 1, 2, \ldots, N-j \tag{4.13}$$

$$b_{N-j,k} = b_{N-j+1,k} - \gamma_j a_{N-j+1,k} \qquad k = 0, 1, 2, \ldots, N-j. \tag{4.14}$$

Just as the sequences $\{a_{N-j+1,k}\}_{k=0}^{N-j+1}$ and $\{b_{N-j+1,k}\}_{k=0}^{N-j+1}$ consist of the same numbers, in opposite orders, so equations (4.13) and (4.14) will produce sequences $\{a_{N-j,k}\}_{k=0}^{N-j}$ and $\{b_{N-j,k}\}_{k=0}^{N-j}$ consisting of the same numbers in opposite orders.

In its simplest terms the algorithm has three steps; we have adapted the procedure of Kailath and Lev-Ari [6]:

I. Take the coefficients of $P_{N-1}(z)$ from equation (4.1) or (4.9) and construct the $2 \times N$ matrix

$$\mathbf{G}_0 = \begin{bmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{bmatrix}.$$

II. Compute $\gamma_2 = a_0/a_{N-1}$ and construct

$$\mathbf{G}_1' = \begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_2 & 1 \end{bmatrix} \mathbf{G}_0 = \begin{bmatrix} 0 & a_0' & a_1' & \cdots & a_{N-3}' & a_{N-2}' \\ a_{N-2}' & a_{N-3}' & & \cdots & a_0' & 0 \end{bmatrix}.$$

III. Shift the top row of the matrix formed in II to the left and delete the last column to form the $2 \times (N-1)$ matrix

$$\mathbf{G}_1 = \begin{bmatrix} a_0' & a_1' & \cdots & a_{N-2}' \\ a_{N-2}' & a_{N-1}' & \cdots & a_0' \end{bmatrix}$$

and go to step I.

## 5. Conclusions

We have shown that Schur's algorithm ensures the existence of, and provides an algorithm for computing, the set of $N$ values of the Goupillaud model of the Sturm–Liouville equation in impedance form. Schur's algorithm is known to be computationally stable and efficient [3].

We note that by making trivial changes in the analysis we can construct the Goupillaud model from $N$ interlacing eigenvalues, $0 < \omega_1 < \omega_2 < \cdots < \omega_N = \pi N/2L$, corresponding to

$$\text{I.} \qquad w'(0) = 0 = w(L) \qquad \text{odd } \omega_j$$

$$\text{II.} \qquad w(0) = 0 = w(L) \qquad \text{even } \omega_j.$$

However, it is *not* possible to use the essentially algebraic method of this paper to construct the $A_i$ from the eigenvalues $0 < \omega_1 < \omega_2 < \cdots < \omega_N$ corresponding to the general end conditions

$$\text{I.} \qquad w'((0) = 0 = w'(L) + Hw(L)$$

$$\text{II.} \qquad w(0) = 0 = w'(L) + Hw(L).$$

This is because $\omega$ will appear in the analysis as itself, and not just in the form $e^{2i\omega\Delta}$.

It is possible also to apply the analysis to the equation

$$w''(x) + \lambda\rho(x)w(x) = 0 \qquad (5.1)$$

which appears in connection with string vibrations, since this may be transformed into (1.1) with an appropriate change of variable.

## References

[1] Andersson L E 1990 Algorithms for solving inverse eigenvalue problems for Sturm–Liouville equations *Inverse Problems in Action* ed P C Sabatier (Berlin: Springer)

[2] Bube K P and Burridge R 1983 The one-dimensional inverse problem of reflection seismology *SIAM Rev.* **25** 497–559

[3] Bruckstein A M and Kailath T 1987 Inverse scattering for discrete transmission-line models *SIAM Rev.* **29** 359–89

[4] Schur J 1917 Über Potenzreihen, die im Innem des Einheitskreises beschränkt sind *J. Reine Angew. Math.* **147** 205–32

[5] Knopp K 1945 *Elements of the Theory of Functions* (New York: Dover)

[6] Kailath T and Lev-Ari H 1985 On mappings between covariance matrices and physical systems *Contemp. Math.* **47** 241–52