Inverse Problems 11 (1995) 533–554. Printed in the UK

On isospectral rods, horns and strings

G M L Gladwell† and A Morassi‡

† Solid Mechanics Division, Faculty of Engineering, University of Waterloo, N2L 3G1, Canada

‡ Istituto di Meccanica Teorica ed Applicata, Universita' degli Studi di Udine, Viale delle Scienze, 33100 Udine, Italy

Received 10 October 1994

Abstract. The free undamped vibrations of rods, horns and taut strings are governed by second-order differential equations. It is known that the inverse problem, namely the reconstruction of such a system, e.g. the reconstruction of the cross-sectional profile of a rod, requires the knowledge of two free vibration spectra corresponding to two different sets of end conditions. This paper is concerned with families of second-order systems which have one spectrum in common. The analysis is based on the reduction of the governing equation to the standard Sturm–Liouville form, the use of the Darboux lemma, and the research of Trubowitz, Pöschel, Deift and others. In particular the paper establishes necessary and sufficient conditions for isospectral flow from one rod to another rod with the same end conditions, using double Darboux transformations.

1. Introduction

The linear, free, undamped longitudinal vibrations of a thin straight elastic rod with variable cross section $A \equiv A(x)$, Young's modulus E and density ρ are governed by the equation

$$\frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) = \rho A\frac{\partial^2 u}{\partial t^2}.$$

For free vibration with frequency ω, the longitudinal displacement $u(x,t)$ may be written

$$u(x,t) = u(x)\cos\omega t$$

so that $u \equiv u(x)$ satisfies

$$(Au')' + \lambda Au = 0 \qquad \lambda = \rho\omega^2/E. \tag{1}$$

With appropriate redefinitions of A, u, λ, this equation governs the modes of vibration of a thin rod in torsion, or of an acoustic horn. We shall phrase our discussion in terms of the rod in longitudinal vibration. After developing the analysis for the rod we will discuss what changes have to be made for the analysis of isospectral taut strings, in section 8.

We assume that the rod has unit length and has elastically restrained ends, so that

$$A(0)u'(0) - ku(0) = 0 = A(1)u'(1) + Ku(1) \tag{2}$$

0266-5611/95/030533+22$19.50 © 1995 IOP Publishing Ltd

where $k, K \geqslant 0$. We distinguish three important cases:

$$\text{Supported (S): } k = \infty = K \qquad u(0) = 0 = u(1) \tag{3}$$

$$\text{Free (F): } k = 0 = K \qquad u'(0) = 0 = u'(1) \tag{4}$$

$$\text{Cantilever (C): } k = \infty \qquad K = 0 \qquad u(0) = 0 = u'(1). \tag{5}$$

The first two are often called Dirichlet and Neumann end conditions, respectively.

There is a fourth case, namely the periodic rod. Here $A(x)$, $u(x)$ have period 1, so that $A(x) = A(1 + x)$, and the end conditions are

$$\text{Periodic (P): } u(0) = u(1) \qquad u'(0) = u'(1). \tag{6}$$

We shall be concerned exclusively with rods for which $A(x)$ is a positive, twice continuously differentiable function of x. (These conditions are unnecessarily restrictive, but we are not interested, at this time, in discussing the fine points of the analysis of the case when A is merely assumed to be, e.g., continuous.) It is well known (see Gladwell 1986, ch 8) that for such $A(x)$, and end conditions (2), there is an infinite sequence $\{\lambda_n\}_0^\infty \equiv \lambda_0, \lambda_1\lambda_2, \ldots$ such that

$$0 \leqslant \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which (1) has a non-trivial solution u satisfying the end conditions (2). Moreover, it is known that $\lambda_0 = 0$ iff the rod is free (F) or periodic (P), and that $\lambda_n \to \infty$ as $n \to \infty$. This sequence is called the spectrum of the rod for the end conditions; we can write

$$\{\lambda_n\}_0^\infty = s(A, k, K). \tag{7}$$

Since equation (1) is homogeneous in A, the spectrum depends only on the shape of A, not on its absolute magnitude, i.e. $s(cA, ck, cK) = s(A, k, K)$. Henceforth we shall assume that A has been normalized so that

$$A(0) = 1. \tag{8}$$

As equation (7) states, the spectrum depends on the shape and the end conditions. For large n, i.e. asymptotically, the λ_n depend more strongly on the end conditions than on the shape. For a non-uniform, but continuous rod, there are three cases:

(i) k, K finite, including the free rod, (F) for which $k = 0 = K$

$$\lambda_n = (n\pi)^2 + \mathrm{O}(1) \tag{9}$$

(ii) one of k, K finite, one infinite, including (C) $k = \infty$, $K = 0$

$$\lambda_n = [(n + 1/2)\pi]^2 + \mathrm{O}(1) \tag{10}$$

(iii) $k = \infty = K$, i.e. (S)

$$\lambda_n = [(n + 1)\pi]^2 + \mathrm{O}(1). \tag{11}$$

In each case the O(1) terms depends on $A(x)$ and the finite k, K, if there is one.

For the periodic case, (P), given by (6)

$$\lambda_n = (2n\pi)^2 + O(1) \tag{12}$$

where the O(1) terms depend on $A(x)$, and are zero when $A(x) = \text{constant}$.

The shape of the rod and the end conditions uniquely determine the spectrum, but, as is well known, the converse is false. In fact two spectra, satisfying specific interlacing conditions and having specific asymptotic forms, and corresponding to two different sets of end conditions, for example (S) and (C), are required to reconstruct the shape and end conditions uniquely. This is well known (see Borg (1946) and Gladwell (1986) for proofs and historical notes) and will be used in section 7. First, however, we are concerned with the problem of finding other rods which have the same spectrum as a given rod for a particular set of end conditions. If

$$s(A_1, k_1, K_1) = s(A_2, k_2, K_2) \tag{13}$$

we shall say that the rods are *isospectral.* The simplest, almost trivial pair of isospectral rods is obtained by physically turning the rod and restraints around, so that

$$A_2(x) = A_1(1-x) \qquad k_2 = K_1 \qquad K_2 = k_1.$$

This will have no effect on the spectrum, so that

$$s(A_1(x), k_1, K_1) = s(A_1(1-x), K_1, k_1). \tag{14}$$

The remainder of the paper runs as follows. In section 2 we derive some simple isospectral pairs. After reducing equation (1) to Sturm–Liouville form in section 3, we introduce the Darboux lemma in section 4 and apply it to form isospectral families of rods in sections 5 and 6. In section 7 we consider isospectral flow from one rod to another, and in section 8 we briefly discuss isospectral strings.

2. Some simple isospectral pairs

To obtain the simplest isospectral pair we note that if u satisfies (1), then $\omega = Au'$ satisfies

$$(A^{-1}\omega')' + \lambda A^{-1}\omega = 0 \tag{15}$$

which is precisely (1) with A replaced by A^{-1}.

Now consider the end conditions. We have

$$\omega = Au' \qquad \omega' = -\lambda A u.$$

Thus if the original rod is a cantilever, with $u(0) = 0 = u'(1)$, then the new rod satisfies $\omega'(0) = 0 = \omega(1)$ so that it is a reversed cantilever. The cantilever cannot have a zero eigenvalue so that we can conclude

$$s(A, \infty, 0) = s(1/A, 0, \infty)$$

and using (14) we deduce that

$$s(A(x), \infty, 0) = s(1/A(1-x), \infty, 0).$$

This is a result which has been known for many years (see Eisner 1967, Benade 1976 or Gladwell 1986 p 149), and was recently pointed out by Ram and Elhay (1994).

If the original rod is free, so that $u'(0) = 0 = u'(1)$, then $\omega(0) = 0 = \omega(1)$, so that the new rod is supported. But the free rod has a zero eigenvalue with eigenfunction $u \equiv 1$, for which $\omega \equiv 0$. Thus the zero eigenvalue will not appear in the spectrum for the supported rod. We conclude that

$$s'(A, 0, 0) = s(1/A, \infty, \infty) \tag{16}$$

where the $'$ indicates that the zero eigenvalue has been omitted.

It is also simple to deduce

$$s(A, \text{periodic}) = s(1/A, \text{periodic}).$$

3. Reduction to Sturm–Liouville form

The cross section function A is positive; write

$$A = a^2 \qquad y = au. \tag{17}$$

Then

$$Au' = a^2u' = ay' - a'y \tag{18}$$

so that (1) reduces to the Sturm–Liouville form

$$y'' + (\lambda - q)y = 0 \tag{19}$$

where

$$q = a''/a. \tag{20}$$

For a given A or a, there is a unique q, but for a given q there are many a. This allows us to obtain further isospectral sets. (Although rather obvious and observed already by Bernoulli and Euler, the indeterminacy introduced by the Liouville transformation in the inverse eigenvalue problem seems to have been systematically studied first by Hochstadt (1975). He proved that classical uniqueness theorems for Sturm–Liouville problem hold, *modulo* a Liouville transformation.) If a_0 is one a corresponding to a given q, then variation of parameters gives the general solution

$$a = a_0\left\{d + b\int_0^x \frac{\mathrm{d}s}{a_0^2(s)}\right\} \qquad b, d \text{ constant.} \tag{21}$$

The normalization condition $a(0) = 1$ gives $d = 1$, so that

$$A = A_0\left\{1 + b\int_0^x \frac{\mathrm{d}s}{A_0(s)}\right\}^2. \tag{22}$$

The constant b must be chosen so that $A > 0$ for $0 \leqslant x \leqslant 1$; a necessary and sufficient condition is

$$\alpha \equiv 1 + b \int_0^1 \frac{\mathrm{d}s}{A_0(s)} > 0. \tag{23}$$

If u_0, u are the solutions of (1) corresponding to exactly the same y equation (19), then

$$a_0 u_0 = y = au.$$

A simple calculation shows that if u_0 satisfies the end conditions (2) with $k = k_0$, $K = K_0$, then u satisfies the end condition with

$$k = k_0 - b \qquad K = K_0\alpha^2 + b\alpha \tag{24}$$

where α is given by (23). Thus, provided that b satisfies

$$-\frac{1}{p} < b < \frac{-K_0}{1 + K_0 p} \qquad p = \int_0^1 \frac{\mathrm{d}s}{A_0(s)} \tag{25}$$

we have a one-parameter isospectral family of rods with

$$s(A, k, K) = s(A_0, k_0, K_0). \tag{26}$$

In particular, if $k_0 = \infty = K_0$, then $k = \infty = K$, and

$$s(A, \infty, \infty) = s(A_0, \infty, \infty)$$

provided only that b satisfies (23).

There are some simple examples of equation (22):

$$A_0(x) = 1 \qquad A(x) = (1 + bx)^2 \tag{27}$$

$$A_0(x) = \mathrm{e}^{2\alpha x} \qquad A(x) = \{\cosh\alpha x + (1 + b/\alpha)\sinh\alpha x\}^2. \tag{28}$$

One particular example of the second is

$$A(x) = \{\cosh\alpha(x - 1/2)/\cosh\alpha/2\}^2 \tag{29}$$

which is symmetrical about the mid-section $x = 1/2$.

4. The Darboux lemma

We showed that the governing equation (1) could be reduced to the Sturm–Liouville equation (19). In a series of papers (Isaacson and Trubowitz 1983, Isaacson *et al* 1984, Dahlberg and Trubowitz 1984) and a book (Pöschel and Trubowitz 1987), Trubowitz and his co-workers have given a complete characterization of the isospectral potentials $q(x)$ for the Sturm–Liouville problem (19) with different sets of fixed and variable boundary conditions. Coleman and McLaughlin (1993a,b) have extended this analysis to equation (1) with Dirichlet boundary conditions. These papers are concerned with a complete characterization of the isospectral sets. The present paper has a more modest aim: to show how to obtain families of rods isospectral to a given one. The analysis used by Trubowitz *et al* is based on a particular case of the factorization result stated earlier, the so-called Darboux lemma.

The Darboux lemma (Darboux 1882, 1915) runs as follows. Let μ be a real number, and suppose $g \equiv g(x)$ is a non-trivial solution of the Sturm–Liouville equation

$$-g'' + \hat{q}g = \mu g \tag{30}$$

with potential $\hat{q} \equiv \hat{q}(x)$. If f is a non-trivial solution of

$$-f'' + \hat{q}f = \lambda f \tag{31}$$

and $\lambda \neq \mu$, then

$$y = \frac{1}{g}[g, f] \equiv \frac{1}{g}(gf' - g'f) \tag{32}$$

is a non-trivial solution of the Sturm–Liouville equation

$$-y'' + \check{q}y = \lambda y \tag{33}$$

where

$$\check{q} = \hat{q} - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln(g(x)). \tag{34}$$

The Darboux lemma enables us to find a solution of a new equation, (33), if we know two solutions g, f of another equation, (30), corresponding to two different values λ, μ of a parameter.

For if $z = gf' - g'f$, then

$$z' = gf'' - g''f = (\mu - \lambda)fg \tag{35}$$

so that z satisfies

$$\left(\frac{z'}{g^2}\right)' = (\mu - \lambda)\frac{z}{g^2} \tag{36}$$

and hence $y = z/g$ satisfies

$$\left(\frac{gy' + g'y}{g^2}\right)' = (\mu - \lambda)\frac{y}{g}$$

i.e.

$$\frac{y''}{g} + \left(\frac{g'}{g}\right)' y = (\mu - \lambda)\frac{y}{g}$$

i.e.

$$-y'' + \left\{\mu - \left(\frac{g'}{g}\right)' g\right\} y = \lambda y$$

i.e.

$$-y''+\left\{\hat{q}-\frac{g''}{g}-\left(\frac{g'}{g}\right)'g\right\}y=\lambda y \tag{37}$$

which is (34).

We note that if f is non-trivial, then y is non-trivial. For if it were trivial, then $z \equiv [g, f]$ would be identically zero, so that $z' = 0$. But then equation (35) yields $(\mu - \lambda)fg = 0$ which is impossible since $\lambda \neq \mu$ and f, g are non-trivial. This establishes the first part of the lemma. There is a second part of the lemma, which states that the general solution of the equation

$$-y''+\check{q}y=\mu y$$

is

$$y=\frac{1}{g}\left(1+c\int_0^x g^2(s)\,\mathrm{d}s\right) \qquad c=\text{constant.}$$

This follows immediately by putting $\lambda = \mu$ in equation (37) and retracing the steps to equation (36) which is now

$$\left(\frac{(yg)'}{g^2}\right)'=0 \qquad \text{i.e. } (yg)'=cg^2.$$

(Pöschel and Trubowitz (1987, p 89) give a longer, but intuitively more instructive derivation of the Darboux lemma in which the Sturm–Liouville operator $-\mathrm{D}^2+\hat{q}-\mu$ is factorized as the product of two operators; by reversing the order of the factors we then obtain the new Sturm–Liouville operator $-\mathrm{D}^2+\check{q}-\mu$.)

5. Single application of the Darboux lemma

Suppose that we have rod $\hat{A} \equiv \hat{A}(x)$ with spectrum $\{\lambda_n\}_0^\infty$ corresponding to end conditions (2). Transforming to Sturm–Liouville form, we have a set of eigenfunctions g_n satisfying

$$-g_n''+\hat{q}g_n=\lambda_n g_n \tag{38}$$

and some end conditions

$$g_n'(0)-hg_n(0)=0=g_n'(1)+Hg_n(1). \tag{39}$$

In particular the zeroth eigenfunction g_0 will satisfy

$$-g_0''+\hat{q}g_0=\lambda_0 g_0. \tag{40}$$

Applying the Darboux lemma, with $g_n, g_0, \lambda_n, \lambda_0$ replacing f, g, λ, μ respectively, we deduce that

$$h_n=\frac{1}{g_0}[g_0, g_n] \tag{41}$$

is a non-trivial solution of

$$-h_n'' + \check{q}h_n = \lambda_n h_n \tag{42}$$

where

$$\check{q} = \hat{q} - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln g_0. \tag{43}$$

We can use the result only if g_0 is positive throughout $0 \leqslant x \leqslant 1$. This will be the case if k, K are finite (see Gladwell 1986). Thus equation (42) will be a proper Sturm–Liouville system. Moreover, since g_n and g_0 both satisfy the same end condition (39), h_n will satisfy

$$h_n(0) = 0 = h_n(1). \tag{44}$$

This means that the eigenfunction of the new Sturm–Liouville equation (42), and hence of any rod corresponding to this potential $\check{q}$, will satisfy the supported end conditions. We must now find a function $\check{a}$ or, in fact, a family of such $\check{a}$, corresponding to the new $\check{q}$.

The original rod with cross section $\hat{A}$ had a function $\hat{a}$ satisfying

$$-\hat{a}'' + \hat{q}\hat{a} = 0.$$

Equation (21) states that all a of the form

$$a = \hat{a}\left\{1 + b\int_0^x \frac{\mathrm{d}s}{\hat{a}^2(s)}\right\} \tag{45}$$

satisfy the same equation, i.e.

$$-a'' + \hat{q}a = 0. \tag{46}$$

Applying the Darboux lemma to (40), (46), we deduce that if $\lambda_0 > 0$, then

$$\check{a} = \frac{1}{g_0}[g_0, a] \tag{47}$$

is a solution of

$$-\check{a}'' + \check{q}\check{a} = 0. \tag{48}$$

For this $\check{a}$ to be a possible $a(x)$ for the new rod with supported ends, it must be of one sign throughout $0 \leqslant x \leqslant 1$. First we note that since $\hat{a} > 0$ throughout $0 \leqslant x \leqslant 1$, the intermediate a given by (45) will be positive throughout if

$$\alpha \equiv 1 + b\int_0^1 \frac{\mathrm{d}s}{\hat{a}^2(s)} \tag{49}$$

is positive, and will have one simple zero if $\alpha \leqslant 0$. Now we show that $\check{a}$ given by (47) can have at most one zero in any interval in which a, given by (45), is of one sign. For suppose $\check{a}$ had two such zeros, x_1, x_2 ($x_1 < x_2$) in such an interval then, by Rolle's theorem, $[g_0, a]'$ must be zero at an intermediate point. But equation (35) shows that

$$[g_0, a]' = \lambda_0 a g_0 \neq 0$$

which is a contradiction.

There are two cases to consider:

(i) a, given by (45) has no zero in $0 \leqslant x \leqslant 1$. Now $a > 0$ and $\alpha > 0$. $\breve{a}$ can have at most one zero in $0 \leqslant x \leqslant 1$, and so will have no zero if it has the same sign at 0 and a. A simple calculation shows that

$$\breve{a}(0) = b - k \qquad \breve{a}(1) = (K\alpha + b)/\hat{a}(1).$$

Thus $\breve{a}$ will have one sign throughout if

$$b > k \qquad \text{or} \qquad -\frac{1}{p} < b < \frac{-K}{1+Kp}$$

where $p = \int_0^1 (1/\hat{a}^2(s))\, \mathrm{d}s$.

(ii) a, given by (45) has one zero in $0 \leqslant x \leqslant 1$. Now $a(\xi) = 0$ for some ξ satisfying $0 \leqslant \xi \leqslant 1$; $\alpha \leqslant 0$ and $b \leqslant -1/p$. Since $\breve{a}(\xi) = b/\hat{a}(\xi) < 0$, $\breve{a}(x)$ will have one sign throughout iff $\breve{a}(0) < 0$, $\breve{a}(1) < 0$, i.e. iff $b < -K/(1 + Kp)$, but since $b \leqslant -1/p$, this is satisfied automatically.

We deduce that equations (45), (47) provide a one-parameter family of isospectral rods if

$$b < \frac{-K}{1+Kp} \qquad \text{or} \qquad b > k. \tag{50}$$

We conclude that provided neither end of the original rod is supported (i.e. k, K are finite) and the rod is not free (i.e. $\lambda_0 > 0$), then the new rod with $\breve{A} = \breve{a}^2$ and the supported end conditions, has the spectrum $\{\lambda_n\}_1^\infty$. Thus

$$s'(A, k, K) = s(\breve{A}, \infty, \infty) \tag{51}$$

where the prime signifies that λ_0 has been deleted.

If the original rod is free, then $\lambda_0 = 0$ and $g_0 = \hat{a}$. Now the second part of the Darboux lemma states that the general solution of equation (48) is

$$\breve{a} = \frac{1}{\hat{a}}\left\{1 + b\int_0^x \hat{a}^2(s)\, \mathrm{d}s\right\} \tag{52}$$

so that

$$s'(\hat{A}, 0, 0) = s(\breve{A}, \infty, \infty). \tag{53}$$

We now show that λ_n and h_n given by (41) are in fact the $(n-1)$th eigenvalue and eigenfunction of the $\breve{A}$ rod. First we show that there is a zero of g_n between two consecutive zeros of h_n. This follows from (35); if x_1, x_2 are two consecutive zeros, then

$$0 = [g_0, g_n]\,|_{x_1}^{x_2} = \int_{x_1}^{x_2} (g_0 g_n'' - g_0'' g_n)\, \mathrm{d}s = (\lambda_0 - \lambda_n)\int_{x_1}^{x_2} g_0 g_n\, \mathrm{d}s.$$

But g_0 has constant sign throughout $0 \leqslant x \leqslant 1$ so that g_n must change sign, and have a zero, between x_1 and x_2. Now we show that there is a zero of h_n between consecutive zeros of g_n. This follows from (41), namely

$$h_n = g_n' - \frac{g_0'}{g_0} g_n$$

when $g_n = 0$, $h_n = g_n'$. But g_n' has opposite signs at successive zeros of g_n. Thus h_n changes sign, and therefore has a zero, between zeros of g_n. We conclude that the zeros of g_n and h_n interlace. But g_n has n zeros in $0 < x < 1$ while $h_n(0) = 0 = h_N(1)$. Therefore h_n has $(n-1)$ zeros in $0 < x < 1$; it is the $(n-1)$th eigenfunction. We may thus rewrite equation (51), (53) respectively as

$$\lambda_n(\hat{A}, k, K) = \lambda_{n-1}(\check{A}, \infty, \infty) \tag{54}$$

$$\lambda_n(\hat{A}, 0, 0) = \lambda_{n-1}(\check{A}, \infty, \infty). \tag{55}$$

We may verify this by considering the various asymptotic forms, for the eigenvalues, as given by Hochstadt (1961) or Gladwell (1986) (but note that, in the latter, ω_n^2 should be replaced by ω_{n-1}^2 in example 8.11.2 on p 184). Thus

$$\left(\lambda_n(\hat{A}, k, K) - \int_0^1 \hat{q}\,\mathrm{d}x\right)^{1/2} = n\pi + \frac{h+H}{n\pi} + \mathrm{O}(n^{-3})$$

$$\left(\lambda_{n-1}(\check{A}, \infty, \infty) - \int_0^1 \check{q}\,\mathrm{d}x\right)^{1/2} = n\pi + \mathrm{O}(n^{-3}).$$

These agree because equations (43), (39) show that

$$\int_0^1 \check{q}\,\mathrm{d}x = \int_0^1 \hat{q}\,\mathrm{d}x - 2\left(\frac{g_0'(1)}{g_0(1)} - \frac{g_0'(0)}{g_0(0)}\right) = \int_0^1 \hat{q}\,\mathrm{d}x + 2(H+h).$$

The simplest example of (55) occurs for the uniform rod. Here $g_0 \equiv 1$ so that $\check{q} = \hat{q}$ and $\check{A} = \hat{A}$; this is evident from equations (9), (10).

The analysis of this section breaks down if g_0 has a zero at an end, as it does when one or other end of the rod is fixed. For such cases we must modify the analysis by applying the Darboux lemma twice, as we shall now describe.

6. Double application of the Darboux lemma

Suppose that we have a rod $\hat{A} = \hat{A}(x)$ with spectrum $\{\lambda_n\}_0^\infty$ corresponding to end conditions (2). Transforming to Sturm–Liouville form, we have a set of eigenfunctions g_n satisfying

$$-g_n'' + \hat{q}g_n = \lambda_n g_n \tag{56}$$

and some end conditions

$$g_n'(0) - hg_n(0) = 0 = g_n'(1) + Hg_n(1) \tag{57}$$

exactly as in equations (38), (39). We now choose a particular eigenvalue and eigenfunction λ_m, g_m; m need not be zero. Thus g_m satisfies

$$-g_m'' + \hat{q}g_m = \lambda_m g_m. \tag{58}$$

Applying the Darboux lemma, we find a non-trivial solution

$$h_n = \frac{1}{g_m}[g_m, g_n] \qquad n \neq m \tag{59}$$

of

$$-h_n'' + \check{q} h_n = \lambda_n h_n \tag{60}$$

where

$$\check{q} = \hat{q} - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln g_m. \tag{61}$$

On the other hand, the second part of the Darboux lemma states that the general solution of

$$-h_m'' + \check{q} h_m = \lambda_m h_m \tag{62}$$

is

$$h_m = \frac{1}{g_m}\left(1 + c\int_0^x g_m^2 \,\mathrm{d}x\right). \tag{63}$$

We now apply the Darboux lemma to equations (60), (62) and deduce that if $n \neq m$, then

$$k_n = \frac{1}{h_m}[h_m, h_n] \tag{64}$$

is a non-trivial solution of

$$-k_n'' + q k_n = \lambda_n k_n \tag{65}$$

where

$$q = \check{q} - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\ln h_m) = \hat{q} - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\ln g_m h_m).$$

We now examine q and the function k_n. first we note that equation (63) gives $g_m h_m$, so that

$$q = \hat{q} - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left[1 + c\int_0^x g_m^2 \,\mathrm{d}s\right]. \tag{66}$$

If g_m has been normalized so that $\int_0^1 g_m^2(s)\,\mathrm{d}s = 1$, then q will be continuous in $0 \leqslant x \leqslant 1$ if $c > -1$. We now evaluate k_n: it is

$$k_n = \frac{1}{h_m}(h_m h_n' - h_m' h_n) = h_n' - h_n \frac{h_m'}{h_m}. \tag{67}$$

But equation (59) shows that

$$h_n' = \frac{g_m g_n'' - g_m'' g_n}{g_m} - h_n \frac{g_m'}{g_m} = (\lambda_m - \lambda_n) g_n - h_n \frac{g_m'}{g_m}$$

so that on substituting this into equation (67) we find

$$k_n = (\lambda_m - \lambda_n)g_n - h_n \frac{\mathrm{d}}{\mathrm{d}x}(g_m h_m)/g_m h_m.$$

But

$$h_n = \frac{g_m g_n' - g_m' g_n}{g_m} = \frac{1}{g_m}\int_0^x (g_m g_n'' - g_m'' g_n)\,\mathrm{d}s$$
$$= \frac{\lambda_m - \lambda_n}{g_m}\int_0^x g_m g_n\,\mathrm{d}s.$$

This means that k_n has a factor $\lambda_m - \lambda_n$, so that if we write

$$k_n^0 = \frac{1}{\lambda_m - \lambda_n} k_n$$

and use equation (63) to give $g_m h_m$, we find

$$k_n^0 = g_n - \frac{c g_m \int_0^x g_m g_n \mathrm{d}s}{1 + c\int_0^x g_m^2 \mathrm{d}s}. \tag{68}$$

Now we see that this is a non-trivial solution of equation (65) *even when n in that equation is equal to m*. It can readily be shown that k_n^0 is normalized so that $\int_0^1 [k_n^0(x)]^2\,\mathrm{d}x = 1$.

To this point we have two Sturm–Liouville equations with potentials $\hat{q}$ and q respectively, and for each solution g_n of the first we have a solution k_n^0 of the second. Moreover, if g_n satisfies $g_n(0) = 0 = g_n(1)$, then k_n^0 satisfies $k_n^0(0) = 0 = k_n^0(1)$, so that $\hat{q}$ and q are isospectral. To examine what happens for other end conditions we must construct the corresponding rods.

To find $a(x)$ we use the Darboux lemma, noting that the original $\hat{a}$, the intermediate $\breve{a}$ and the final a are solutions of the equations

$$-\hat{a}'' + \hat{q}\hat{a} = 0 \tag{69}$$

$$-\breve{a}'' + \breve{q}\breve{a} = 0 \tag{70}$$

$$-a'' + qa = 0 \tag{71}$$

respectively. Thus the Darboux lemma applied to equations (58), (69) shows that a non-trivial $\breve{a}$ is

$$\breve{a} = \frac{1}{g_m}[g_m, \hat{a}]$$

while the Darboux lemma applied to equations (62), (70) shows that a non-trivial a is

$$a = \frac{1}{h_m}[h_m, \breve{a}]. \tag{72}$$

Note that this $\breve{a}$ and a are just one of each of the families of $\breve{a}$'s and a's corresponding to $\breve{q}$ and q respectively; all others may be found by using the analysis of section 3. We can find a just as we found k_n:

$$a = \hat{a} - \frac{c[g_m, \hat{a}]g_m}{\lambda_m(1 + c\int_0^x g_m^2(s)\,\mathrm{d}s)}. \tag{73}$$

We see immediately that if the original rod has cross sectional $\hat{A} = \hat{a}^2$ and satisfies the supported end conditions, and if a, $\hat{a}$ are linked by equation (73), then the new rod $A = a^2$ with the supported end conditions, has the same spectrum, i.e.

$$s(\hat{A}, \infty, \infty) = s(A, \infty, \infty). \tag{74}$$

We must now examine other end conditions.

Suppose the end conditions for the original rod $\hat{A}$ are

$$\hat{A}(0)u'(0) - \hat{k}u(0) = 0 = \hat{A}(1)u'(1) + \hat{K}u(1). \tag{75}$$

Equations (17), (18) show that these transform to

$$[\hat{a}, g_n]_0 - \hat{k}g_n(0)/\hat{a}(0) = 0 = [\hat{a}, g_n]_1 + \hat{K}g_n(1)/\hat{a}(1) \tag{76}$$

so that the end values of $a(x)$ given by (73) satisfy

$$\frac{a(0)}{\hat{a}(0)} = 1 - \frac{c[g_m, \hat{a}]_0 g_m(0)}{\lambda_m[\hat{a}(0)]} = 1 + \frac{c\hat{k}g_m^2(0)}{\lambda_m[\hat{a}(0)]^2} = \beta_0 \tag{77}$$

$$\frac{a(1)}{\hat{a}(1)} = 1 - \frac{c[g_m, \hat{a}]_1 g_m(1)}{\lambda_m[\hat{a}(1)](1+c)} = 1 - \frac{c\hat{K}g_m^2(1)}{\lambda_m[\hat{a}(1)]^2(1+c)} = \beta_1. \tag{78}$$

Note that unless the left-hand end of the original rod is supported or free, the new $a(x)$ will not be normalized so that $a(0) = 1$. We first show that if $c > -1$ then β_0, β_1 are both positive. Let $u_m = g_m/\hat{a}$ be the mth mode of the rod. Then $-(\hat{A}u')' = \lambda_m \hat{A} u_m$ so that

$$\int_0^1 u_m(\hat{A}u_m')' \, dx = -[u_m \hat{A} u_m']_0^1 + \int_0^1 \hat{A}[u_m']^2 \, dx = \lambda \int_0^1 \hat{A}u_m^2 \, dx = \lambda_m \int_0^1 g_m^2 \, dx = \lambda_m$$

and on using the end conditions (75) we find

$$\lambda_m > \hat{K}u_m^2(1) + \hat{k}u_m^2(0)$$

so that

$$\beta_0 > \frac{\hat{k}u_m^2(0)(1+c) + \hat{K}u_m^2(1)}{\lambda_m} > 0$$

$$\beta_1 > \frac{\hat{k}u_m^2(0)(1+c) + \hat{K}u_m^2(1)}{\lambda_m(1+c)} > 0.$$

These inequalities will hold provided that at least one end of the rod is not supported, i.e. either $u_m(0) \neq 0$ or $u_m(1) \neq 0$. Thus the only case which is excluded is the supported rod (S), given by (3).

We now have a one-parameter family of rods $a(x) \equiv a_c(x)$ defined for $0 \leqslant x \leqslant 1$ and $c > -1$; each member of the family is positive at $x = 0$, $x = 1$, and when $c = 0$, $a_0(x) \equiv \hat{a}(x)$ is positive in $0 \leqslant x \leqslant 1$. To show that $a_c(x)$ must be positive in $0 \leqslant x \leqslant 1$ for all $c > -1$ we use the following *deformation lemma*:

Lemma 1. Let h_c, $c_1 \leqslant c \leqslant c_2$, be a family of real valued functions of x in $[0, 1]$ which is jointly continuously differentiable in c and in x. Suppose that for every c, h_c has a finite number of zeros in $[0, 1]$, all of which are simple, and has boundary values with signs that are independent of c. Then, the number of zeros of h_c is independent of c, for all c satisfying $c_1 \leqslant c \leqslant c_2$.

This is a slightly extended version of a lemma derived by Pöschel and Trubowitz (1987, p 41); they simply supposed that '*h_c has boundary values that are independent of c*'; however, their proof may easily be extended to yield the required result.

It may easily be seen that $a(x)$ can have only a finite number of roots, and that these must be simple, so that the lemma implies that $a(x)$ must be positive for $0 \leqslant x \leqslant 1$ and $c > -1$. The deformation lemma above can be used to show that a given by (73) is strictly positive in the case of the supported rod (S) also.

We now examine the end conditions for the new rod. The eigenfunctions of the new rod are v_n, where $av_n = k_n^0$.

A tedious, but straightforward calculation shows that, at $x = 0$ and at $x = 1$,

$$k_n^0 = \gamma g_n \qquad [a, k_n^0] = \gamma[\hat{a}, g_n]$$

where

$$\gamma = \begin{cases} 1/(1+c) & \text{when } n = m \text{ and } x = 1 \\ 1 & \text{otherwise.} \end{cases} \tag{79}$$

This means that the end conditions for the new rod have constants k, K, where

$$k = \beta_0 \hat{k} \qquad K = \beta_1 \hat{K}. \tag{80}$$

Thus

$$s(\hat{A}, \hat{k}, \hat{K}) = s(A, k, K) \tag{81}$$

and, in particular,

$$s(\hat{A}, 0, 0) = s(A, 0, 0) \tag{82}$$

$$s(\hat{A}, \infty, 0) = s(A, \infty, 0). \tag{83}$$

It must be remembered, of course, that the particular A which is formed from a given $\hat{A}$ depends on the end conditions corresponding to $\hat{A}$, so that the A's in equations (81), (82), (83) are all different.

7. Isospectral flow

Two rods with their cross sectional profiles $A_1(x)$, $A_2(x)$ and their corresponding end conditions specified by k_1, K_1 and k_2, K_2 are said to be isospectral if they have the same spectrum, i.e. if equation (13) holds. We now ask whether it is possible to flow from one rod to another by means of a finite or infinite sequence of double Darboux transformations (DDTs).

We base our answer to this question on the result, due to Borg (1946) and Levinson (1949), that if the spectrum of the Sturm–Liouville equation (17) is known for two sets of end conditions (39), with the same value of h, and two different values H_1, H_2 (one must therefore be finite) of H, then $q(x)$ is uniquely determined. But knowing the spectra for two such sets of end conditions is equivalent to knowing the spectrum for one set (39), with H finite, and the values $\{y_n(1)\}_0^\infty$ of the normalized eigenfunctions at $x = 1$. We can transfer this result to the rod and can deduce that $A(x)$ is uniquely determined by its value $A(0)$, its spectrum $\{\lambda_n\}_0^\infty$ for the end conditions (2), with K finite, and the end values $\{u_n(x)\}_0^\infty$ of the eigenfunctions normalized so that

$$\int_0^1 A(x)[u_n(x)]^2 \, dx = 1.$$

This means that, when K is finite, we can label any such rod by the sequence $\{u_n(1)\}_0^\infty$ of normalized eigenfunction values.

If two rods are isospectral, so that they have the same spectrum, they must necessarily have the same asymptotic spectrum. The asymptotic spectrum is given by equations (9)–(11). This means that if two rods are isospectral then *either* k_1, K_1, k_2, K_2 are all finite *or* one of k_1, K_1 and one of k_2, K_2 are infinite *or* both k_1, K_1 and k_2, K_2 are infinite.

We can think of a DDT as an operator $\mathcal{D}_{m,c}$ specified by a non-negative integer m and a real parameter $c > -1$. The operator $\mathcal{D}_{m,c}$ changes $\hat{a}$ into a given by equation (73); g_n into k_n^0 according to equation (68); and $\hat{k}$, $\hat{K}$ into $k = \beta_0\hat{k}$, $K = \beta_1\hat{K}$ where β_0, β_1 are given by (77), (78). We can deduce how $\mathcal{D}_{m,c}$ changes the mode shape u_n into v_n by noting that

$$\hat{a}u_n = g_n \qquad av_n = k_n^0.$$

Thus in particular

$$\frac{v_n(1)}{u_n(1)} = \frac{k_n^0(1)}{g_n(1)}\frac{\hat{a}(1)}{a(1)} = \gamma\frac{1}{\beta}$$

where γ is given by equation (79).

By turning the rod around, if necessary, we can reduce our study to six cases: (i) (C) $k = \infty$, $K = 0$, (ii) (F) $k = 0 = K$, (iii) (S) $k = \infty = K$, (iv) $0 < k < \infty$, $K = 0$, (v) $k = \infty$, $0 < K < \infty$ and (vi) $0 < k < \infty$, $0 < K < \infty$. Equations (77), (78) show that β_0, β_1 are unity iff k, K take the values 0 or ∞, i.e. in cases (i)–(iii).

(i) (C) i.e. $k = \infty$, $K = 0$. Under $\mathcal{D}_{m,c}$, $a(0) = \hat{a}(0)$ and

$$\frac{v_n(1)}{u_n(1)} = 1 \qquad n \neq m \qquad \frac{v_m(1)}{u_m(1)} = \frac{1}{1+c}.$$

Thus any cantilever rod specified by $\{u_n(1)\}_0^\infty$ may flow to a cantilever specified by $\{v_n(1)\}_0^\infty$ by the sequence of transformations

$$\prod_{m=0}^{\infty} \mathcal{D}_{m,c_m} \qquad \frac{v_m(1)}{u_m(1)} = \frac{1}{1+c_m}.$$

We note that the transformation $\prod_{m=0}^N \mathcal{D}_{m,c_m}$ will enforce the required end values $\{v_m(1)\}_0^N$; and the asymptotic results $|u_n(1)| \to 1$, $|v_n(1)| \to 1$ mean that $c_n \to 0$ as $n \to \infty$. Strictly, it is necessary to examine the effect on $a(x)$ on an infinite sequence of such transformations.

(ii) (F) i.e. $k = 0$, K. We may treat this essentially the same way; we may flow from any free rod to any other free rod.

(iii) (S) $k = \infty = K$. To treat this we use equation (16). For any supported rod $A_1(x)$ there is a free rod $B_1(x) = 1/A_1(x)$, such that

$$s'(B_1, 0, 0) = s(A_1, \infty, \infty).$$

From $B_1(1)$ we can pass to any isospectral free rod $B_2(x)$ as described above, and thence to $A_2(x) = 1/B_2(x)$ via

$$s'(B_2, 0, 0) = s(A_2, \infty, \infty).$$

These cases are the only ones in which any rod can flow to any other isospectral rod. In fact, as we shall show, any rod cannot even flow to any other isospectral rod with the same end conditions.

We begin our consideration of cases (iv)–(vi) by noting what happens to $a(x)$, $u_n(x)$, k, K after successive operations $\mathcal{D}_{m,c}$ with the same m and different values of c, namely $c_m^{(1)}, c_m^{(2)}, \ldots, c_m^{(N)}$. Changing our notation, we start from $a_0(x)$, $u_n^{(0)}(x)$, k_0, K_0 and suppose

$$\mathcal{D}_{m,c_m^{(i)}}\{a_{i-1}(x), u_n^{(i-1)}(x), k_{i-1}, K_{i-1}\} = \{a_i(x), u_n^{(i)}(x), k_i, K_i\} \qquad i = 1, 2, \ldots, N.$$

Equations (77)–(79) show that

$$\frac{a_1(0)}{a_0(0)} = 1 + \frac{c_m^{(1)} k_0 g_m^2(0)}{\lambda_m [a_0(0)]^2} = \beta_0^{(1)} \qquad \frac{a_2(0)}{a_1(0)} = 1 + \frac{c_m^{(2)} (\beta_0^{(1)} k_0) g_m^2(0)}{\lambda_m [\beta_0^{(1)} a_0(0)]^2} = \beta_0^{(2)}$$

so that on writing

$$\frac{k_0 g_m^2(0)}{\lambda_m [a_0(0)]^2} = \delta_m$$

we find

$$\frac{a_2(0)}{a_0(0)} = \left(1 + \frac{c_m^{(2)} \delta_m}{\beta_0^{(1)}}\right) \beta_0^{(1)} = 1 + \delta_m (c_m^{(1)} + c_m^{(2)})$$

and generally

$$\frac{a_N(0)}{a_0(0)} = \frac{k_N}{k_0} = 1 + \delta_m \sum_{i=1}^{N} c_m^{(i)}. \tag{84}$$

A similar calculation shows that

$$\frac{a_N(1)}{a_0(1)} = \frac{K_N}{K_0} = 1 - \epsilon_m \sum_{i=1}^{N} \frac{c_m^{(i)}}{1 + c_m^{(i)}} \prod_{j=1}^{i-1} \frac{1}{(1 + c_m^{(j)})^2} \tag{85}$$

where

$$\epsilon_m = \frac{K_0 g_m^2(1)}{\lambda_m [a_0(1)]^2}.$$

Note that, unlike the sum in (84), the sum in (85) depends on the order in which the $c_m^{(i)}$ are taken. We may now treat (iv).

(iv) $0 < k < \infty$, $K = 0$. Now $a_N(1) = a_0(1)$ so that equation (79) shows that, of the normalized eigenmode values at $x = 1$, only the mth is changed, and

$$u_m^{(N)}(1) = u_m^{(0)}(1) \prod_{i=1}^{N} \frac{1}{(1 + c_m^{(i)})}.$$

If the end condition at $x = 0$ is to remain invariant, then equation (84) shows that we must choose the $c_m^{(i)} > -1$ so that

$$\sum_{i=1}^{N} c_m^{(i)} = 0. \tag{86}$$

If we are to choose

$$u_m^{(N)}(1)/u_m^{(0)}(1) = r_m \tag{87}$$

then we must take

$$\prod_{i=1}^{N} (1 + c_m^{(i)}) = \frac{1}{r_m}. \tag{88}$$

Put $1 + c_m^{(i)} = x_i$, then $x_i > 0$, and equations (86), (88) become

$$\sum_{i=1}^{N} x_i = N \qquad \prod_{i=1}^{N} x_i = \frac{1}{r_m}.$$

But since the x_i are positive and the geometric mean of $x_1, x_2, \ldots, x_N$ is not greater than the arithmetic mean, we must have

$$(x_1, x_2, \ldots, x_N)^{1/N} \leqslant \frac{x_1 + x_2 + \cdots + x_N}{N} = 1 \tag{89}$$

so that $r_m \geqslant 1$. (There is equality in (89) iff all the x_i are equal; since their arithmetic mean is 1, they must then all be 1, i.e. $c_m^{(i)} = 0$, $i = 1, 2, \ldots, N$.) This means that it is possible to flow from one system $0 < k < \infty$, $K = 0$ to another with the same end conditions only if

$$r_m \geqslant 1 \qquad m = 0, 1, 2, \ldots. \tag{90}$$

(Note that we can, and do, choose $u_m^{(0)} > 0$.) If (90) holds, then we can take $N = 2$, so that

$$c_m^{(1)} + c_m^{(2)} = 0 \qquad (1 + c_m^{(1)})(1 + c_m^{(2)}) = 1/r_m$$

i.e.

$$c_m^{(1)} = -c_m^{(2)} = \sqrt{1 - 1/r_m}.$$

The next case is (v) $k = \infty$, $0 < K < \infty$. If the end condition at $x = 1$ is to remain invariant we must choose the $c_m^{(i)}$ so that the sum in (85) is zero, and so that (87) is satisfied, i.e.

$$\sum_{i=1}^{N} \frac{c_m^{(i)}}{1 + c_m^{(i)}} \prod_{j=1}^{i-1} \frac{1}{(1 + c_m^{(j)})^2} = 0 \qquad \prod_{i=1}^{N} (1 + c_m^{(i)}) = \frac{1}{r_m}.$$

Putting $c_m^{(i)} = -1 + 1/u_i$, $r_m = r$, we find

$$F \equiv \sum_{i=1}^{N} (1 - u_i) \prod_{j=1}^{i-1} u_j^2 = 0 \qquad \prod_{i=1}^{N} u_i = r. \tag{91}$$

After some tedious calculation we find that

$$F \geqslant \frac{(1-r)}{2N} \{N + 1 + (N-1)r\}$$

so that when $0 < r < 1$, there is no positive solution of (91). When $r > 1$ we may take $N = 2$ so that

$$1 - u_1 + (1 - u_2)u_1^2 = 0 \qquad u_1 u_2 = r$$

i.e. $1 - (r + 1)u_1 + u_1^2 = 0$, which has positive roots.

The final case is (vi) $0 < k < \infty$, $0 < K < \infty$. If the end conditions are to remain invariant, and if (87) is to hold we must choose $c_m^{(i)} = -1 + 1/u_i$ so that

$$\sum_{i=1}^{N} \frac{1}{u_i} = N \qquad \sum_{i=1}^{N} (1 - u_i) \prod_{j=1}^{i-1} u_j^2 = 0 \qquad \prod_{i=1}^{N} u_i = r.$$

Again these have a positive solution only if $r \geqslant 1$. When $r > 1$ we may take $N = 3$ and, after some tedious routine calculation, show that the equations have a positive solution.

8. Isospectral strings

The free vibration of a string with linear density $\rho \equiv \rho(x)$ pulled with tension T is governed by

$$T \frac{\partial^2 u}{\partial x^2} = \rho(x) \frac{\partial^2 u}{\partial t^2}.$$

For free vibration with frequency ω, $u(x, t)$ may be written

$$u(x, t) = u(x) \cos \omega t$$

so that $u \equiv u(x)$ satisfies

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\omega^2}{T} \rho(x) u(x) = 0. \tag{92}$$

If the ends of the strings are restrained with springs of stiffnesses k, K, then

$$T\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_0 - ku(0) = 0 = T\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_l + Ku(l). \tag{93}$$

If $\rho(x)$ is twice continuously differentiable, we may make the Liouville transformation

$$\xi = \frac{1}{p}\int_0^x [\rho(s)]^{1/2}\,\mathrm{d}s \qquad p = \int_0^l [\rho(s)]^{1/2}\,\mathrm{d}s$$

$$\rho(x) = \frac{p^2}{l^2}a^4(\xi) \qquad y(\xi) = a(\xi)u(x)$$

to reduce equation (92) to the dimensionless equation

$$y'' + (\lambda - q)y = 0 \qquad \lambda = p^2\omega^2/T \tag{94}$$

over $0 \leqslant \xi \leqslant 1$. Here $' \equiv \mathrm{d}/\mathrm{d}\xi$, and

$$q(\xi) = a''(\xi)/a(\xi). \tag{95}$$

The end conditions (93) transform into

$$\frac{T}{l}[a, y]_0 - k\left(\frac{y}{a}\right)_0 = 0 = \frac{T}{l}[a, y]_1 + K\left(\frac{y}{a}\right)_1.$$

The first observation we can make concerning isospectral strings stems from the asymptotic form of the eigenvalues. Whatever the end conditions, and whatever the potential q, the eigenvalues λ_n of (94) have the asymptotic form

$$\lambda_n = n\pi + \mathrm{O}(1).$$

Therefore if two strings with parameters $\rho_1(x)$, l_1, T_1 and $\rho_2(x)$, l_2, T_2 are isospectral, i.e. $(\omega_n)_1 = (\omega_n)_2$, then, for large n,

$$\frac{p_1^2(\omega_n)_1^2}{T_1} = \lambda_n = n\pi + \mathrm{O}(1) = \frac{p_2^2(\omega_n)_2^2}{T_2}$$

so that

$$\frac{p_1^2}{T_1} = \frac{p_2^2}{T_2}. \tag{96}$$

The analysis of isospectral strings follows exactly the same lines as that for isospectral rods, except that, after having found $a(\xi)$ from $q(\xi)$ using equation (95), we must find the revised independent variable x from

$$x = l\int_0^{\xi} \frac{\mathrm{d}\xi}{a^2(\xi)}. \tag{97}$$

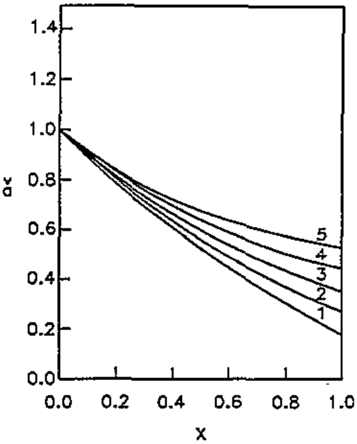

Figure 1. Single application of the Darboux lemma: examples of rods under supported end conditions with the same ('higher') spectrum as a given rod (with $\hat{A}(x) = e^{2x}$) under free–free end conditions. Curves 1–5 for $b = -0.15, -0.075, 0, 0.075, 0.15$; $\check{a}(x)$ as in equation (52).

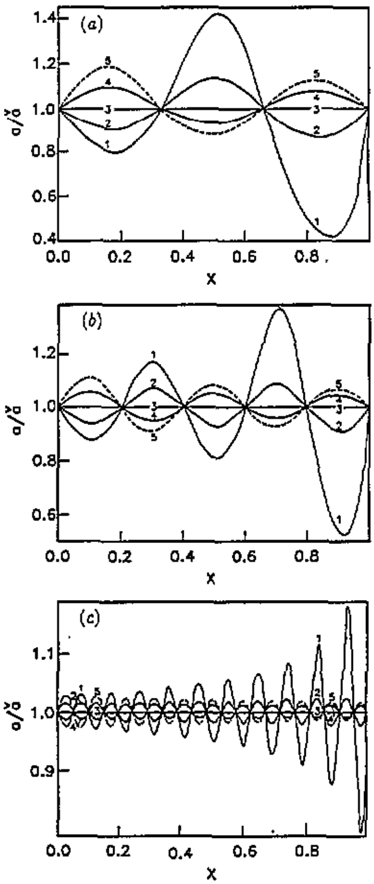

Figure 2. Double application of the Darboux lemma: examples of isospectral rods under supported end conditions with $\hat{A} = 1$, $a(x)$ as in equation (73) and $c = -0.9, -0.45, 0, 0.45, 0.9$ (curves 1–5 respectively): (a) $m = 1$; (b) $m = 2$; (c) $m = 10$.

9. Examples

Figure 1 shows a pair of rods related by a single application of the Darboux lemma. Here the initial rod is free, and has $\hat{A}(x) = e^{2x}$. The derived rod

$$\check{A}(x) = \frac{1}{\hat{A}(x)}\left\{1 + c\int_0^x \hat{A}(s)\,ds\right\}^2$$

has the same supported spectrum as the first except for the rigid-body mode. We note that, when $c = 0$,

$$\check{A}(x) = \frac{1}{\hat{A}(x)} = \frac{\hat{A}(1-x)}{\hat{A}(1)}.$$

Since the conditions at the ends are the same, the supported spectrum for $\hat{A}(x)$ is the same as that for $\hat{A}(1-x)$, i.e. for $\check{A}(x)$ when $c = 0$.

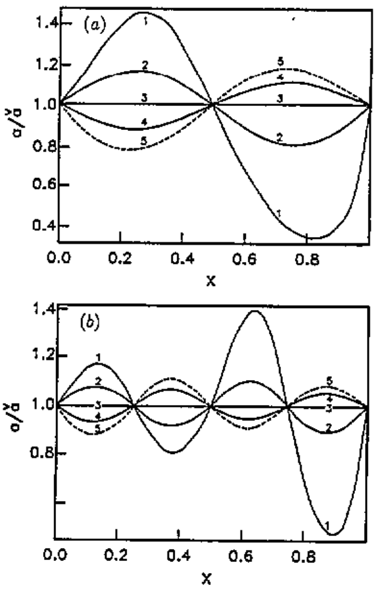

Figure 3. Double application of the Darboux lemma: examples of isospectral rods under free–free end conditions with $\hat{A} = 1$, $a(x)$ as in equation (73) and $c = -0.9, -0.45, 0, 0.45, 0.9$ (curves 1–5 respectively): (*a*) $m = 1$; (*b*) $m = 2$.

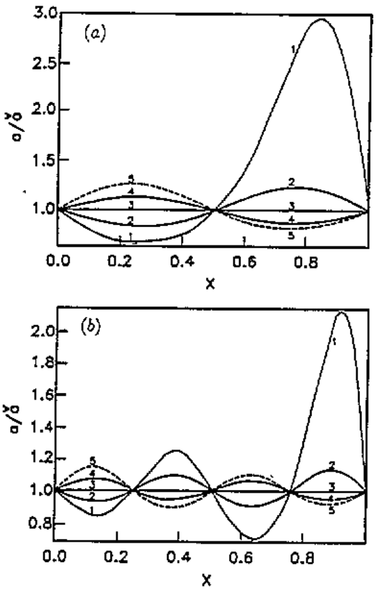

Figure 4. Double application of the Darboux lemma: examples of isospectral rods under cantilever end conditions with $\hat{A} = 1$, $a(x)$ as in equation (73) and $c = -0.9, -0.45, 0, 0.45, 0.9$ (curves 1–5 respectively): (*a*) $m = 1$; (*b*) $m = 2$.

Figures 2–4 show families of rods which are isospectral to the uniform rod $\hat{A}(x) = 1$ for supported, free and cantilever end conditions respectively. These have been derived using equation (73) with $m = 1, 2$ and 10. We note that when c is taken towards the limit -1, or m is taken to be large, then the profile departs significantly from the uniform rod, so that we could not expect its vibrations to be adequately described by the simple equation (1); transverse or other motions would have to be considered.

10. Conclusions

We have shown that if we start with one rod and one set of end conditions, we may construct many families of rods which have the same spectrum as the original for specific sets of end conditions. The indeterminacy arises from the Liouville transformation to normal form or from one or more applications of the Darboux lemma. In each case the new rods and their normal modes can be constructed explicitly, i.e. without extra computation.

We showed that for the three extreme end conditions, (S), (C), (F) listed in section 1 any rod can flow to any isospectral rod with the same end conditions. Under the three remaining sets of end conditions, listed (iv)–(vi) in section 7, a rod can flow to an isospectral rod with the same end conditions only iff r_m defined by equation (87) is greater than or equal to unity for $m = 0, 1, 2, \ldots$.

The paper therefore leaves open the question of whether there is some other kind of transformation, apart from the double Darboux transformation, by which isospectral flow is possible when $r_m < 1$.

References

Benade A H 1976 *Fundamentals of Musical Acoustics* (London: Oxford University Press)

Borg G 1946 Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. *Acta Math.* 78 1–96

Coleman C F and McLaughlin J R 1993 Solution of the inverse spectral problem for an impedance with integrable derivative. I *Comm. Pure Appl. Math.* **46** 145–84

—— 1993 II *Comm. Pure Appl. Math.* **46** 185–212

Dahlberg B E J and Trubowitz E 1984 The inverse Sturm–Liouville problem. III *Comm. Pure Appl. Math.* **37** 255–67

Darboux G 1882 Sur la représentations sphérique des surfaces *C. R. Acad. Sci., Paris* **94** 1343–5

—— 1915 *Leçons sur le Théorie Générale des Surfaces et les Applications Géométrique du Calcul Infinitesimal* vol 2 (Paris: Gauthier Villars)

Deift P 1978 Applications of a commutation formula *Duke Math. J.* **45** 267–310

Eisner E 1967 Complete solution of the 'Webster' horn equation *J. Acoust. Soc. Am.* **41** 1126–46

Gladwell G M L 1986 *Inverse Problems in Vibration* (Dordrecht: Kluwer)

Hochstadt H 1961 Asymptotic estimates of the Sturm–Liouville spectrum *Comm. Pure Appl. Math.* **14** 749–64

—— 1975 On inverse problems associated with Sturm–Liouville operators *J. Diff. Eqns* **17** 220–35

Isaacson E L and Trubowitz E 1983 The inverse Sturm–Liouville problem. I *Comm. Pure Appl. Math.* **36** 767–83

Isaacson E L, McKean H P and Trubowitz E 1984 The inverse Sturm–Liouville problem. II *Comm. Pure Appl. Math.* **37** 1–11

Pöschel J and Trubowitz E 1987 *Inverse Spectral Theory* (London: Academic)

Ram Y M and Elhay S 1994 Dualities in vibrating rods and beams: continuous and discrete models *J. Sound Vib.* (to appear)