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# The total positivity interval

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**Abstract**

If $A \in M_n$ is totally positive (TP), we determine the maximum open interval $\mathscr{I}$ around the origin such that, if $\mu \in \mathscr{I}$, then $A - \mu I$ is TP. If $A$ is TP, $\mu \in \mathscr{I}$ and $A - \mu I = LU$, then $B$ defined by $B - \mu I = UL$ is TP, and has the same total positivity interval $\mathscr{I}$. If $A$ is merely nonsingular and totally nonnegative (TN), or oscillatory, there need be no such interval in which $A - \mu I$ is TN.


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## 1. Introduction

Totally positive, and the related terms totally nonnegative and oscillatory, are important descriptors in the characterization of matrices appearing in a variety of contexts, see Gantmakher and Krein [3], Gladwell [5].

A matrix $A \in M_n$ is said to be *totally positive* (TP) *(totally nonnegative* (TN)) if every minor of $A$ is positive (nonnegative). It is NTN if it is invertible and TN. It is *oscillatory* (O) if it is TN and a power of $A$, $A^m$, is TP. If $Z = \mathrm{diag}(+1, -1, +1, \ldots)$ and $ZAZ$ is O, then $A$ is said to be *sign oscillatory* (SO); sign oscillatory is a particular case of *sign regular.*

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Cryer [2] proved that if $A$ is NTN, then it has a unique factorization *LU* with $L$ lower triangular and having unit diagonal, $U$ upper triangular, and $B = UL$ is also NTN. We may extend Cryer's result to matrices that are TP, O or SO. If $A$ is TP then so is $B$. If $A$ is O then it is NTN, so $B$ is NTN. A power of $A$ is TP so that $A^m = (LU)^m$ is TP, and then $B^{m+1} = (UL)^{m+1} = U(LU)^m L$ is TP; $B$ is O. Similarly if $A$ is SO, so is $B$.

For symmetric $A$, i.e., $A \in S_n$, Gladwell [6] extended Cryer's result as follows. Let $P$ denote one of the properties TP, NTN, O or SO. If $A$ has property P, $\mu$ is not an eigenvalue of $A$, $A - \mu I = QR$ where $Q$ is orthogonal and $R$ is upper triangular with diagonals chosen to be positive, and $B$ is defined by $B - \mu I = RQ$, then $B$ also has property P. This result depends on the fact that if $A \in S_n$ and is nonsingular, then its *QR* factorization may be effected by making two successive *LU* factorizations. This is not true for general $A \in M_n$.

The following counterexample, with $\mu = 0$, shows that Gladwell's result can not be extended to general $A \in M_n$:

$$A = \begin{bmatrix} 2 & a \\ 1 & 2 \end{bmatrix}, \ Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \ R = \sqrt{5} \begin{bmatrix} 1 & (2a+2)/5 \\ 0 & (4-a)/5 \end{bmatrix},$$

$$B = \frac{1}{5} \begin{bmatrix} 12+2a & 4a-1 \\ 4-a & 2(4-a) \end{bmatrix}.$$

If $a = 1/5$, then $A$ is TP, but $B$ is not even TN. In Section 2 we find a restricted version of Gladwell's result that holds for $A \in M_n$.

## 2. The total positivity interval

The *TP-interval* of a TP matrix, denoted by $\mathscr{I}_A$, is the maximum open interval around zero such that $A - \mu I$ is TP for $\mu \in \mathscr{I}_A$. We seek this interval.

Following Ando [1] we let $Q_{p,n}$ denote the set of strictly increasing sequences of $p$ integers taken from $\{1, 2, \ldots, n\}$. If $\alpha = (\alpha_1, \ldots, \alpha_p) \in Q_{p,n}$ and $\beta = (\beta_1, \ldots, \beta_p) \in Q_{q,n}$, we denote the submatrix of $A$ lying in rows indexed by $\alpha$ and columns indexed by $\beta$, by $A[\alpha|\beta]$.

When $\alpha \in Q_{p,n}$ and $\beta \in Q_{q,n}$, and $\alpha \cap \beta = \phi$, then $\alpha \cup \beta$ is rearranged increasingly to become a member of $Q_{p+q,n}$.

We use Sylvester's identity on bordered determinants:

If $\alpha, \beta \in Q_{p,n}$ let $C = (c_{ij})$ where $c_{ij} = \det A[\alpha \cup i|\beta \cup j]$, and $\gamma, \delta \in Q_{q,n}$, then

$$\det C[\gamma|\delta] = (\det A[\alpha|\beta])^{q-1} \det A[\alpha \cup \gamma|\beta \cup \delta].$$

This states that if the minors of $A$ are positive, then the minors of $C$ are positive also. This matrix $C$ is bordered about the submatrix $A[\alpha|\beta]$.

**Theorem 1.** *Suppose $A \in M_n$ is TP. Then $\mathscr{I}_A = (-a, b)$ where a and b are defined in* (5).

**Proof.** The *corner minors* of $A$ are $\det A[1, 2, \ldots, p|n-p+1, \ldots, n]$ and det $A[n-p+1, \ldots, n|1, 2, \ldots, p]$ for $p = 1, 2, \ldots, n$. It is known (Gasca and Pena [4], Gladwell [6]) that if $A$ is TN and its corner minors are strictly positive, then $A$ is TP. We may use this result to narrow the search for the total positivity interval. Consider what happens to the minors of $A - \mu I$ as $\mu$ increases (decreases) from zero. Suppose if possible that one or more non-corner minors are the first to become zero, at $\mu = \mu_0$. At $\mu_0$, $A - \mu I$ is TN but its corner minors are strictly positive; $A$ is TP, contradicting the assumption that a minor is zero. Thus we may seek the interval in which the corner minors are positive. We examine these corner minors. Let $m = [n/2]$, the integral part of $n/2$. Consider the corner minors taken from the top right corner; these are $\det(A - \mu I)[1, 2, \ldots, p|n-p+1, \ldots, n]$, $p = 1, 2, \ldots, n$. For $p = 1, 2, \ldots, m$ the corner minors are independent of $\mu$; for $p = m+1, \ldots, n$ the variable $\mu$ appears in the diagonal terms $a_{i,i} - \mu$. We may partition the pth order submatrix, and write its determinant as follows:

$$\Delta_p(\mu) = \left|\begin{array}{ccc|ccc} a_{q,q+1} & \cdots & q_{1,p} & q_{1,p+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{q,q+1} & \cdots & a_{q,p} & a_{q,p+1} & \cdots & a_{q,n} \\ \hline a_{q+1,q+1} - \mu & & & a_{q+1,p+1} & \cdots & a_{q+1,n} \\ & \ddots & & \vdots & \cdots & \vdots \\ & & a_{p,p-\mu} & a_{p,p+1} & \cdots & a_{p,n} \end{array}\right| \tag{1}$$

Here $q = n - p$. Now use Sylvester's identity. If $C_p = (c_{ij})$.

$$\begin{aligned} c_p &= \det A[1, 2, \ldots, q|p+1, \ldots, n] \\ c_{ij} &= \det A[1, 2, \ldots, q, i|j, p+1, \ldots, n], \quad i, j = q+1, \ldots, p \end{aligned}$$

then, after taking account of the change of sign arising from the column interchanges, we find

$$c_p^{p-q-1} \Delta_p(\mu) = \det(C_p + (-)^{q-1} c_p \mu I_{p-q}). \tag{2}$$

Sylvester's identity shows that $C_p \in M_{p-q}$ is TP, so that all the eigenvalues of $C_p$ are positive. If $q$ is odd then (2) shows that

$$\Delta_p(\mu) > 0 \quad \text{if } c_p \mu > -\lambda_{p,\lim,R} \tag{3}$$

where $\lambda_{p,\lim,R}$ is the least eigenvalue of $C_p$; $R$ denotes the fact that we are considering right-hand corner minors. If $q$ is even then (2) shows that

$$\Delta_p(\mu) > 0 \quad \text{if } c_p \mu < \lambda_{p,\lim,R}. \tag{4}$$

For each odd $q$, (3) gives a lower bound for $\mu$; for each even $q$, (4) gives an upper bound for $\mu$. The TP interval $\mathscr{I}_A = (-a, b)$ is bounded by the least of these upper bounds and the greatest of the lower bounds. Thus

$$a = \min_{p=m+1,\dots,n;q\ odd} c_p^{-1}\{\lambda_{p,min,R}, \lambda_{p,min,L}\},$$
$$b = \min_{p=m+1,\dots,n;q\ even} c_p^{-1}\{\lambda_{p,min,R}, \lambda_{p,min,L}\} \quad (5)$$

where $L$ denotes the eigenvalues derived from the left-hand corner minors. □

Numerical experiments indicated that there was no particular ordering among the eigenvalues $\lambda_{p,min}$ for different values of $p$. It proved to be difficult to find a TP matrix $A$ such that $A - \mu I$ loses its total positivity for a positive value of $\mu$ less than that $\lambda_1$, the lowest eigenvalue of $A$. However, for the TP matrix

$$A = \begin{bmatrix} 1.8756 & 0.7300 & 1.2706 & 11.7002 & 8.1829 \\ 1.8747 & 0.7513 & 1.3534 & 12.5589 & 8.9982 \\ 1.8003 & 0.7433 & 1.3884 & 12.9948 & 9.5591 \\ 1.6674 & 0.7070 & 1.3636 & 12.8930 & 9.7773 \\ 1.4929 & 0.6492 & 1.2889 & 12.3143 & 9.6265 \end{bmatrix}$$

the top right 3×3 minor of $A - \mu I$ becomes zero at $\mu = 4.1190e^{-05}$, which is less than $\lambda_1 = .0001$, the lowest eigenvalue of $A$; this shows that $A - \mu I$ can lose its total positivity for values of $\mu$ such that $0 < \mu < \lambda_1$.

**Corollary 2.1.** *If $\mathscr{I}_A = (-a, b)$ is the TP interval for $A$, then $A - diag(\mu_1, \mu_2, \dots, \mu_n)$ is TP provided that $\mu_i \in \mathscr{I}_A$, $i = 1, 2, \dots, n$.*

**Proof.** Let

$$\mu_r = \max_{i=1,2,\dots,n} \mu_i, \quad \mu_s = \min_{i=1,2,\dots,n} \mu_i.$$

Consider the minor $\Delta_p$ $(\mu_1, \mu_2, \dots, \mu_n)$ obtained by replacing $\mu I_{p-q}$ by $\text{diag}(\mu_{q+1}, \dots, \mu_p)$ in (1). If $q$ has even parity

$$\begin{aligned} c_p^{p-q-1}\Delta_p(\mu_1, \mu_2, \dots, \mu_n) &= \det(C_p - c_p\text{diag}(\mu_{q+1}, \dots, \mu_p)) \\ &= \det\{C_p - c_p\mu_r I_{p-q} + c_p\text{diag}((\mu_r - \mu_{q+1}), \dots, (\mu_r - \mu_p))\} \\ &\geqslant \det(C_p - c_p\mu_r I_{p-q}) = c_p^{p-q+1}\Delta_p(\mu_r) > 0 \end{aligned}$$

because all the minors of $C_p - c_p\mu_r I_{p-q}$ are positive. Similarly, if $q$ has odd parity, then

$$\Delta_p(\mu_1, \mu_2, \dots, \mu_n) \geqslant \Delta_p(\mu_s) > 0. \quad \square$$

**Theorem 2.** *If $A \in M_n$ is TP and $A$ has LU factorization $A = LU$ where $L$ has unit diagonal, then $B = UL$ has the same TP interval $\mathscr{I}$ as $A : \mathscr{I}_A = \mathscr{I}_B$.*

**Proof.** For given $p$, denote the matrix $C_p$ and scalar $c_p$ for $B$ by $D_p$, $d_p$ respectively

$$\begin{aligned} c_p &= \det A[1, 2, \dots, q|p+1, \dots, n] = \det U[1, 2, \dots, q|p+1, \dots, n] \\ &= \det B[1, 2, \dots, q|p+1, \dots, n] = d_p \end{aligned}$$

and

$$C_p = L[q+1, \ldots, p]V[q+1, \ldots, p]$$

where

$$v_{ij} = \det U[1, 2, \ldots, q, i|j, p+1, \ldots, n], \quad i, j = 1+1, \ldots, p$$

while the corresponding matrix obtained from $B$ is

$$D_p = V[q+1, \ldots, p]L[q+1, \ldots, p].$$

But $C_p$ and $D_p$ have the same eigenvalues, so that each upper (lower) bound appearing in (5) for $A$ appears also in the corresponding bound for $B$, and *vice versa*. Hence $B$ has TP interval $\mathscr{I}_A$. □

**Corollary 2.2.** *Suppose* $A \in M_n$ *is TP,* $\nu \in \mathscr{I}_A$, $A - \nu I = LU$, $B - \nu I = UL$, *then* $B$ *is TP with TP interval* $\mathscr{I}_A$.

**Proof.** If $A$ has TP interval $\mathscr{I}_A = (-a, b)$, then $A - \nu I$ has TP interval $(-a - \nu, b - \nu)$; $B - \nu I$ has TP interval $(-a - \nu, b - \nu)$; $B$ has TP interval $(-a, b)$. □

It appears that it is not possible to extend Theorem 1 to matrices that are merely TN or NTN, as the following examples show. The matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is TN, but $A - \mu I$ is not TN for any $\mu \neq 0$.

Now we seek $A \in M_n$ that is NTN but which has no interval around zero in which it is NTN. Take $n = 5$, so that

$$A - \mu I = \begin{bmatrix} a_{11} - \mu & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} - \mu & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} - \mu & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} - \mu & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} - \mu \end{bmatrix}$$

Write $G = A - \mu I$. Consider the two corner minors $\det G[1, 2, 3|3, 4, 5]$ and $\det G[2, 3, 4, 5|1, 2, 3, 4]$. We neek to make the former negative for all positive $\mu$, and the latter negative for all negative $\mu$. To do this, we need to make

$$\det A[1, 2, 3|3, 4, 5] = 0, \quad \det A[1, 2|4, 5] > 0$$

$$\det A[2, 3, 4, 5|1, 2, 3, 4] = 0, \quad a_{51} > 0.$$

Factorize $A = LU$; these conditions will be satisfied if we can find $L$, $U$, both NTN, such that

$$\det U[1, 2, 3|3, 4, 5] = 0, \quad \det U[1, 2|4, 5] > 0$$

$$\det L[2, 3, 4, 5|1, 2, 3, 4] = 0, \quad l_{51} > 0.$$

These conditions are satisfied by

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 8 & 12 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 9 & 13 \\ 1 & 2 & 6 & 12 & 18 \\ 1 & 2 & 6 & 13 & 21 \\ 1 & 2 & 6 & 13 & 22 \end{bmatrix}.$$

Now we have

$$\det G[1, 2, 3|3, 4, 5] = -4\mu, \quad \det G[2, 3, 4, 5|1, 2, 3, 4] = \mu^3.$$

Note that $A$ is not just NTN, it is O. This counterexample shows that even an oscillatory matrix need not have an interval $\mathscr{I}_A$ in which it is TN.

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