On the reconstruction of a damped vibrating system from two complex spectra I; theory

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Abstract:

Paper concerns an n-degree of freedom damped vibrating system consisting of $n-1$ masses connected in parallel, by springs and dampers, to an nth mass. Paper analyses the construction of such a system from given complex eigenvalue data. The analysis has two parts: the establishment of the conditions on the eigenvalues which ensure that they correspond to an actual system; the derivation of the system parameters from the eigenvalues.

1 *INTRODUCTION*

This paper concerns a linear vibrating system, with n degrees of freedom, governed by an equation of the form

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t) \tag{1}$$

where $\cdot = d/dt$. The substitutions

$$\mathbf{u}(t) = \mathbf{u}e^{\lambda t} \qquad \mathbf{f}(t) = \mathbf{f}e^{\lambda t} \tag{2}$$

lead to the equation

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{u} = \mathbf{f}. \tag{3}$$

We will consider the case in which $\mathbf{M}, \mathbf{C}, \mathbf{K}$ are symmetric, $\mathbf{M}$ and $\mathbf{K}$ positive-definite (p-d) and $\mathbf{C}$ positive semi-definite (ps-d).

The free vibration of the system is governed by the equation

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{u} = \mathbf{0}. \tag{4}$$

The values of λ for which this equation has a non-trivial solution, form the *complex spectrum* of the *quadratic pencil*

$$\mathbf{Q}(\lambda) = \mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K}. \tag{5}$$

The values of λ in the spectrum appear in pairs, n pairs in all: real negative pairs, corresponding to *overdamped* modes; complex conjugate pairs corresponding to *underdamped* modes; or complex conjugate imaginary pairs corresponding to *undamped* modes.

We consider a *second spectrum* of (5), consisting of these values of λ for which (4) has a non-trivial solution having $u_n = 0$. This spectrum will consist of $(n-1)$ pairs, again of the three possible kinds. This spectrum is that for the *truncated* pencil

$$Q_L(\lambda) = \mathbf{M}_L\lambda^2 + \mathbf{C}_L\lambda + \mathbf{K}_L \tag{6}$$

where $\mathbf{M}_L,\ \mathbf{C}_L,\ \mathbf{K}_L$ are obtained from $\mathbf{M},\ \mathbf{C},\ \mathbf{K}$ respectively by deleting the nth row and column of each of the three matrices.

The $2n-1$ pairs of eigenvalues contained in the spectra of $\mathbf{Q}(\lambda)$ and $\mathbf{Q}_L(\lambda)$ are clearly insufficient to determine the $3n(n+1)/2$ coefficients in the three matrices $\mathbf{M},\ \mathbf{C},\ \mathbf{K}$. To obtain a unique solution, or a manageable family of solutions, we must drastically constrain the form of the matrices. Figures 1 and 2 show two possible systems. That shown in Figure 1 is an in-line set of masses $(m_i)_1^n$ connected by springs $(k_i)_1^n$ and dampers $(c_i)_1^n$; that shown in Figure 2 is a parallel system of masses $(m_i)_1^{n-1}$ all connected both to ground and to a mass m_n, by springs and dampers.

The outline of the paper is as follows. In Section 2, we summarize the known results for the undamped version of the system in Figure 1; the results for the damped version are given in Section 3. Then in Sections 4, 5 we analyse the undamped and damped versions of the parallel system shown in Figure 2.

2 *THE UNDAMPED SERIES SYSTEM*

For an undamped system, each spectrum consists of complex conjugate imaginary pairs; $\pm i\omega_j,\ j = 1, 2, \ldots, n$, for the unconstrained system; $\pm i\sigma_j,\ j = 1, 2, \ldots, n-1$ for the constrainted. For this case, it is convenient to write

$$\lambda_j = \omega_j^2, \quad \mu_j = \sigma_j^2. \tag{7}$$

Since the μ_j are the squares of the natural frequencies of the constrained system, they must satisfy the interfacing conditions

$$0 \le \lambda_1 \le \mu_1 \le \lambda_2 \le \ldots \le \mu_{n-1} \le \lambda_n. \tag{8}$$

If the system is grounded, i.e., $k_1 > 0$, and connected, i.e., $(k_i)_2^n > 0$, then all the inequalities in (8) are strict, i.e.,

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \ldots < \mu_{n-1} < \lambda_n. \tag{9}$$

We will consider only this case.

The time-reduced equation governing the free vibrations of the undamped system is

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = \mathbf{0}, \tag{10}$$

where

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & \\ -k_2 & k_2 + k_3 & -k_3 & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ & & -k_n & k_n \end{bmatrix}, \mathbf{M} = diag(m_1, m_2, \ldots, m_n). \tag{11}$$

The substitutions

$$\mathbf{M} = \mathbf{M}^{\frac{1}{2}} \cdot \mathbf{M}^{\frac{1}{2}}, \quad \mathbf{M}^{\frac{1}{2}}\mathbf{u} = \mathbf{x}, \quad \mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}} = \mathbf{A} \tag{12}$$

reduce (10) to the standard form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{13}$$

The matrix $\mathbf{A}$ is symmetric, tridiagonal, with negative co-diagonal.

Gantmakher and Krein [1] first solved the basic problem of reconstructing $\mathbf{A}$ from the $(2n-1)$ quantities $(\lambda_j)_1^n$ and $(\mu_j)_1^{n-1}$. Golub and Boley [2] gave a stable numerical algorithm for constructing $\mathbf{A}$. In the vast literature related to the problem and its generalizations, for which see Gladwell [3, 4, 5], we mention Gladwell and Willms [6], Ram and Gladwell [7] and Ram [8]. Gladwell [9] showed how to construct an isospectral family of (undamped) systems like that in (10) which had just one given spectrum $(\lambda_j)_1^n$. Gladwell [10] generalized the problem to finite element method (FEM) models with tridiagonal, rather than simply diagonal, mass matrix.

3 *THE DAMPED SERIES SYSTEM*

The quadratic pencil corresponding to the damped series system of Figure 1 is (5), where $\mathbf{K}, \mathbf{M}$ are given by (11) and

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & & \\ -c_2 & c_2 + c_3 & -c_3 & \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \\ & & -c_{n-1} & c_n \end{bmatrix}. \tag{14}$$

The substitutions

$$\mathbf{M} = \mathbf{M}^{\frac{1}{2}} \cdot \mathbf{M}^{\frac{1}{2}}, \quad \mathbf{M}^{\frac{1}{2}}\mathbf{u} = \mathbf{x}, \quad \mathbf{M}^{-\frac{1}{2}}\mathbf{C}\mathbf{M}^{-\frac{1}{2}} = \mathbf{B}, \quad \mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}} = \mathbf{A} \tag{15}$$

reduces the equation (4) to

$$(I\lambda^2 + \mathbf{B}\lambda + \mathbf{A})\mathbf{x} = \mathbf{0}. \tag{16}$$

Now both $\mathbf{A}, \mathbf{B}$ are symmetric, tridiagonal, p-d and ps-d matrices, respectively. Ram and Elhay [11] studied the reconstruction of $\mathbf{A}$ and $\mathbf{B}$ from two spectra. The basic step in their reconstruction is a generalization of the basic step used in some of the early papers, e.g., Hald [12], in the undamped case, as we now describe.

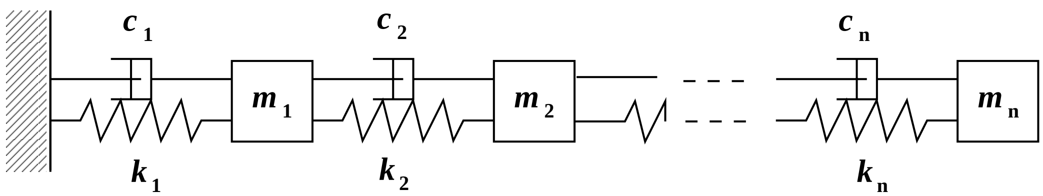

Figure 1: A series system of in-line damped vibrators.

In the undamped case there is just one matrix $\mathbf{A}$, and it has the form

$$\mathbf{A} = \begin{bmatrix} a_1 & -b_1 & & \\ -b_1 & a_2 & -b_2 & \\ \cdots & \cdots & \cdots\cdots\cdots & \\ & & -b_{n-1} & a_n \end{bmatrix}. \tag{17}$$

The principal minors $P_r(\lambda)$ of $\mathbf{A} - \lambda\mathbf{I}$ form a Sturm sequence with initial values

$$P_0(\lambda) = 1, \quad P_1(\lambda) = a_1 - \lambda, \tag{18}$$

and recurrence relation

$$P_r(\lambda) = (a_r - \lambda)P_{r-1}(\lambda) - b_{r-1}^2 P_{r-2}(\lambda). \tag{19}$$

The given data, $(\lambda_j)_1^n$ and $(\mu_j)_i^{n-1}$ are the zeros of $P_n(\lambda)$ and $P_{n-1}(\lambda)$ respectively. Thus

$$P_n(\lambda) = \prod_{j=1}^{n}(\lambda_j - \lambda), \quad P_{n-1}(\lambda) = \prod_{j=1}^{n-1}(\mu_j - \lambda). \tag{20}$$

Now by considering (19) with $r = n$, and knowing $P_n(\lambda)$ and $P_{n-1}(\lambda)$, we can find $P_{n-2}(\lambda)$, a_n and b_{n-1} by synthetic division. Having found $P_{n-2}(\lambda)$, we repeat this step to find successively $a_{n-1}, b_{n-2}; \ldots ; a_2, b_1; a_1$.

The generalization to the damped case is as follows. The matrix $\mathbf{B}$ of equation (16) has the form

$$\mathbf{B} = \begin{bmatrix} d_1 & -e_1 & & \\ -e_1 & d_2 & -e_2 & \\ \cdots & \cdots & \cdots\cdots\cdots & \\ & & -e_{n-1} & d_n \end{bmatrix}. \tag{21}$$

The polynomials corresponding to the $P_1(\lambda)$ of (18) are

$$P_0(\lambda) = 1, \quad P_1(\lambda) = \lambda^2 + d_1\lambda + a_1, \tag{22}$$

and the recurrence relation is now

$$P_r(\lambda) = (\lambda^2 + d_r\lambda + a_r)P_{r-1}(\lambda) - (e_{r-1}\lambda + b_{r-1})^2 P_{r-2}(\lambda). \tag{23}$$

Again the given data $(\lambda_j)_1^{2n}$ and $(\mu_j)_1^{2n-2}$ (which appear as complex conjugate or real pairs, as we noted earlier) are the zeros of $P_n(\lambda)$ and $P_{n-1}(\lambda)$, i.e.,

$$P_n(\lambda) = \prod_{j=1}^{2n}(\lambda_j - \lambda), \quad P_{n-1}(\lambda) = \prod_{j=1}^{2n-2}(\mu_j - \lambda). \tag{24}$$

The essential contribution which Ram and Elhay made was an algorithm to carry out the synthetic division needed to compute $a_r, d_r, e_{r-1}, b_{r-1}$ and $P_{r-2}(\lambda)$ from $P_r(\lambda)$ and $P_{r-1}(\lambda)$. The main difficulty which they encounter is that of not knowing the conditions which the two complex spectra must satisfy to ensure that the pencil is real and the matrices p-d or ps-d. We will encounter this difficulty in our analysis given below, but it will occur in a somewhat milder form.

4 *THE UNDAMPED PARALLEL SYSTEM*

We suppose that two spectra $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ are given, and that they satisfy the strict interlacing condition (9). We construct a system, shown in Figure 2, which has these two spectra. The equation

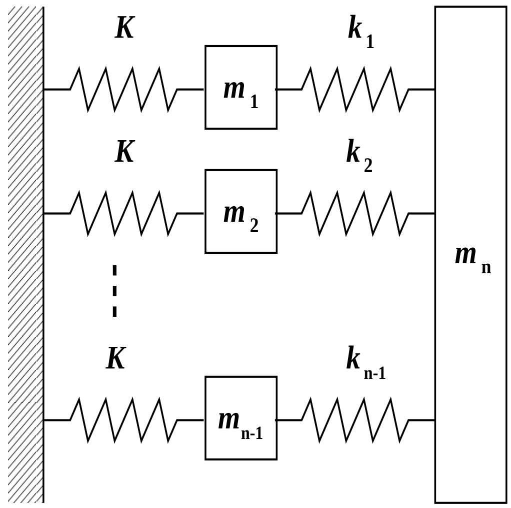

Figure 2: An undamped parallel system of vibratiors.

(10) has the form

$$\begin{bmatrix} K+k_1-m_1\lambda & & & -k_1 \\ & K+k_2-m_2\lambda & & -k_2 \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ & & K+k_{n-1}-m_{n-1}\lambda & -k_{n-1} \\ -k_1 & -k_2 & -k_{n-1} & k-m_n\lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_{n-1} \\ u_n \end{bmatrix} = \mathbf{0}, \qquad (25)$$

where

$$k = \sum_{j=1}^{n-1} k_j. \qquad (26)$$

The eigenvalues of the constrained system are

$$\mu_j = (K+k_j)/m_j, \;\; j = 1, 2, \ldots, n-1. \qquad (27)$$

When reduced to standard form, (25) is

$$(A-\lambda I)x \equiv \begin{bmatrix} \mu_1-\lambda & & & -b_1 \\ & \mu_2-\lambda & & -b_2 \\ \cdots & \cdots & \cdots & \cdots \\ & & \mu_{n-1}-\lambda & -b_{n-1} \\ -b_1 & -b_2 & -b_{n-1} & a_n-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ x_n \end{bmatrix} = \mathbf{0}, \qquad (28)$$

where

$$b_j = k_j/(m_j m_n)^{\frac{1}{2}}, \quad a_n = k/m_n. \tag{29}$$

The trace of the reduced matrix $\mathbf{A}$ is

$$\sum_{j=1}^{n-1} \mu_j + a_n = \sum_{j=1}^{n} \lambda_j, \tag{30}$$

which yields a_n. Now the first $(n-1)$ lines of (28) give

$$(\mu_j - \lambda)x_j = b_j x_n \tag{31}$$

which, when substituted into the last line, gives the eigenvalue equation

$$f(\lambda) \equiv -\sum_{j=1}^{n-1} \frac{b_j^2}{\mu_j - \lambda} + a_n - \lambda = 0 \tag{32}$$

The zeros and poles of $f(\lambda)$ are $(\lambda_j)_1^n$ and $(\mu_j)_1^{n-1}$, so that

$$f(\lambda) = \prod_{j=1}^{n}(\lambda_j - \lambda) / \prod_{j=1}^{n-1}(\mu_j - \lambda) \tag{33}$$

and hence

$$-b_j^2 = \prod_{i=1}^{n}(\lambda_i - \mu_j) / \prod_{i=1}^{n-1\prime}(\mu_i - \mu_j) \tag{34}$$

where $\prime$ denotes $i \neq j$. The strict interlacing condition (9) implies $b_j^2 > 0$.

We have now identified the reduced matrix $\mathbf{A}$; the first $(n-1)$ diagonal elements are data, the last is given by (30); the bordering elements are given by (34).

Now we must construct the masses and stiffnesses; we do that by reversing equations (26), (27) and (29). Put $m_1 = 1,\ m_j = y_j^2,\ j = 1, 2, \ldots, n-1$, then (29) gives $k_j = b_j y_j$. Then equation (27), and equation (26) and (29), respectively, give

$$\mu_j y_j^2 - b_j y_j - K = 0, \quad j = 1, 2, \ldots, n-1, \tag{35}$$

$$-\sum_{j=1}^{n-1} b_j y_j + a_n = 0. \tag{36}$$

Equation (35) is a quadratic equation for y_j with just one positive root:

$$y_j = \frac{b_j + \sqrt{b_j^2 + 4K\mu_j}}{2\mu_j} \tag{37}$$

When substituted into (36), this yields an equation for K:

$$g(K) \equiv -\sum_{j=1}^{n} b_j \frac{\left\{b_j + \sqrt{b_j^2 + 4K\mu_j}\right\}}{2\mu_j} + a_n = 0. \tag{38}$$

The function $g(K)$ is monotonically decreasing, $g(K) \to \infty$ as $K \to \infty$, and $g(0) = f(0)$, where $f(\lambda)$ is given by (32). Now $f(\mu_1-) < 0$ and $f(\lambda_1) = 0$, $0 < \lambda_1 < \mu_1$, imply $f(0) > 0$. Thus $g(K)$ has just one positive root K. Having found K we may find y_j from (37) and complete the reconstruction of the system: $k_j = b_j y_j, m_j = y_j^2$.

We have now constructed a unique system of the form shown in Figure 2, which has the desired spectra. This unique system is a particular member of the family shown in Figure 3. For given

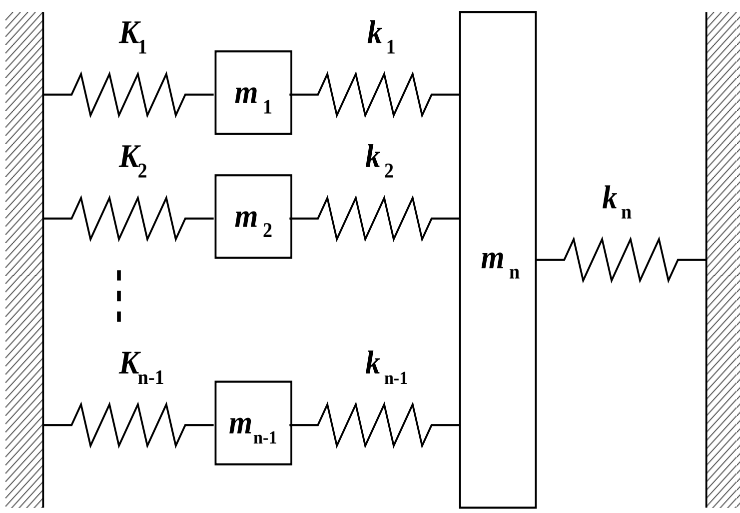

Figure 3: A more general undamped parallel system of n vibrators

reduced matrix shown in equation (28), the equations corresponding to equations (35), (36) are now

$$\mu_j y_j^2 - b_j y_j - K_j = 0, \quad j = 1, 2, \ldots, n-1, \tag{39}$$

$$-\sum_{j=1}^{n-1} b_j y_j + a_n = k_n. \tag{40}$$

Here $(\mu_j, b_j)_1^{n-1}$ and a_n are known and $(y_j, K_j)_1^{n-1}$ and k_n are unknown. Equation (39) gives

$$y_j = \frac{b_j + \sqrt{b_j^2 + 4\mu_j K_j}}{2\mu_j}, \tag{41}$$

and thus equation (40) becomes

$$-\sum_{j=1}^{n-1} b_j \frac{\left(b_j + \sqrt{b_j^2 + 4\mu_j K_j}\right)}{2\mu_j} + a_n = k_n. \tag{42}$$

The left hand side is a function $g(K_1, K_2, \ldots, K_{n-1})$ and, as before $g(0, 0, \ldots 0) > 0$. Since g is continuous at $(0, 0, \ldots, 0)$, there is a neighbourhood of $(0, 0, \ldots, 0)$ in which $g(K_1, \ldots, K_{n-1}) > 0$. Thus there always exists an infinite family of solutions to (42) corresponding to positive $(K_j)_1^{n-1}$ and positive k_n. Once we have chosen one such solution we may form $(y_j)_1^{n-1}$ from (41) and then $(k_j, m_j)_1^{n-1}$ as before.

5 *THE DAMPED PARALLEL SYSTEM*

We consider the system shown in Figure 4. The equation (4) is

$$\begin{bmatrix} F_1(\lambda) & & & & -(g_1\lambda + k_1) \\ & F_2(\lambda) & & & -(g_2\lambda + k_2) \\ & & \ddots & & \vdots \\ & & & F_{n-1}(\lambda) & -(g_{n-1}\lambda + k_{n-1}) \\ -(g_1\lambda + k_1) & -(g_2\lambda + k_2) & \ldots & -(g_{n-1}\lambda + k_{n-1}) & F_n(\lambda) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{43}$$

where

$$\left.\begin{aligned} F_j(\lambda) &= m_j\lambda^2 + (g_j + G_j)\lambda + k_j + K_j \qquad j = 1, 2, \ldots, n-1 \\ F_n(\lambda) &= m_n\lambda^2 + g\lambda + k, \quad g = \textstyle\sum_{j=1}^n g_j, \quad k = \sum_{j=1}^n k_j \end{aligned}\right\} \tag{44}$$

We suppose that the system is connected, so that $(k_j)_1^n > 0$, $(K_j)_1^n > 0$. We now reduce (43) to standard form

$$\mathbf{Q}(\lambda)\mathbf{x} = \mathbf{0} \tag{45}$$

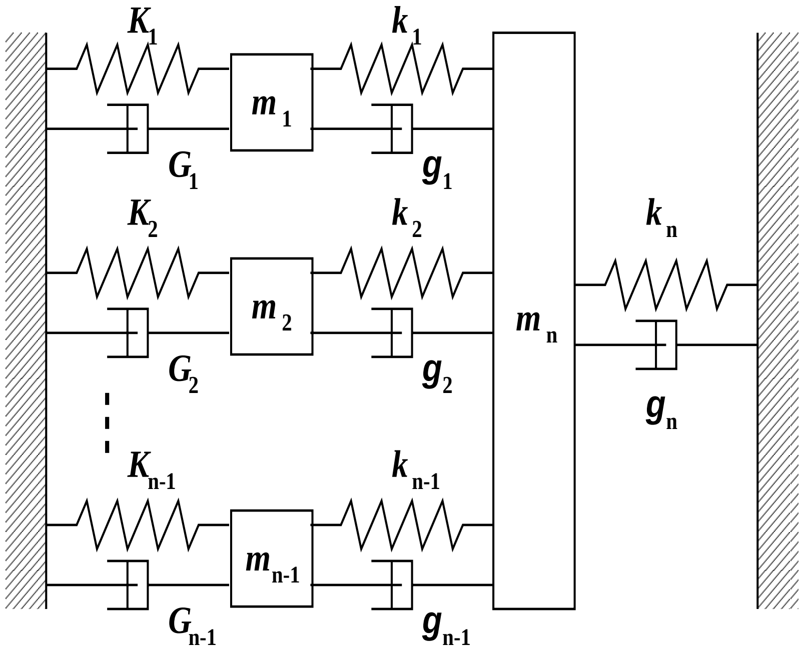

Figure 4: A damped parallel system of n vibrators

where

$$\mathbf{Q}(\lambda) = \begin{bmatrix} \mathbf{Q}_L(\lambda) & -\mathbf{a}\lambda - \mathbf{b} \\ -\mathbf{a}\lambda - \mathbf{b} & \lambda^2 + c_n\lambda + \sigma_n^2 \end{bmatrix} = \lambda^2\mathbf{I}_n + \lambda\mathbf{A} + \mathbf{B} \tag{46}$$

$$\mathbf{Q}_L(\lambda) = \lambda^2\mathbf{I}_{n-1} + \lambda\mathbf{C} + \sum^2 \tag{47}$$

$$\mathbf{C} = \ diag(c_1, c_2, \ldots, c_{n-1}), \quad \sum^2 = \ diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_{n-1}^2) \tag{48}$$

$$c_j = \frac{g_j + G_j}{m_j}, \ \sigma_j^2 = \frac{k_j + K_j}{m_j}, \ a_j = \frac{g_j}{(m_j m_n)^{\frac{1}{2}}}, \ b_j = \frac{k_j}{(m_j m_n)^{\frac{1}{2}}}, \quad j = 1, 2, \ldots, n-1 \tag{49}$$

$$c_n = \frac{g}{m_n}, \ \sigma_n^2 = \frac{k}{m_n}. \tag{50}$$

The $2n$ eigenvalues of the real symmetric quadratic pencil $\mathbf{Q}(\lambda)$ occur in pairs, either real or complex conjugate. The physically important case is when they all lie in the left hand half of the complex plane. We assume therefore that there are $r(0 \leq r \leq n)$ real pairs λ_j, λ_j' where $\lambda_j < 0, \lambda_j' < 0, j = 1, 2, \ldots, r$, and $n-r$ complex conjugate pairs $\lambda_j, \overline{\lambda}_j$, where $Re(\lambda_j) \leq 0, \ Im(\lambda_j) >$

$0, \quad j = r+1, \ldots, n$. The first r pairs correspond to overdamped modes, and the remainder to undamped ($Re(\lambda_j) = 0$) or underdamped ($Re(\lambda_j) < 0$) modes.

We label the eigenvalues of $\mathbf{Q}_L(\lambda)$ similarly: μ_j, μ_j', where $\mu_j < 0$, $\mu_j' < 0$, $j = 1, 2, \ldots, s$; $\mu_j, \overline{\mu}_j$, where $Re(\mu_j) \leq 0, \; Im(\mu_j) > 0, \; j = s+1, \ldots, n-1$.

We now choose $(c_j, \sigma_j^2)_1^{n-1}$ so that

$$\lambda^2 + c_j\lambda + \sigma_j^2 = \left\{ \begin{array}{ll} (\lambda - \mu_j)(\lambda - \mu_j'), & j = 1, 2, \ldots, s, \\ (\lambda - \mu_j)(\lambda - \overline{\mu}_j), & j = s+1, \ldots, n-1. \end{array} \right\} \tag{51}$$

This means that $\mathbf{Q}_L(\lambda)$ has the specified eigenvlues.

Now we must choose the remaining quantities $(a_j, b_j)_1^{n-1}, \; c_n$ and σ_n^2 so that

$$det(\mathbf{Q}(\lambda)) = \prod_{j=1}^{n}(\lambda^2 + d_j\lambda + \omega_j^2), \tag{52}$$

where

$$\lambda^2 + d_j\lambda + \omega_j^2 = \left\{ \begin{array}{ll} (\lambda - \lambda_j)(\lambda - \lambda_j'), & j = 1, 2, \ldots, r, \\ (\lambda - \lambda_j)(\lambda - \overline{\lambda}_j), & j = r+1, \ldots, n. \end{array} \right\} \tag{53}$$

The determinant of $\mathbf{Q}(\lambda)$, as given by (46), is

$$det(\mathbf{Q}(\lambda)) = \prod_{j=1}^{n}(\lambda^2 + c_j\lambda + \sigma_j^2) - \sum_{j=1}^{n-1}(a_j\lambda + b_j)^2 \prod_{k=1}^{n-1\,\prime} (\lambda^2 + c_k\lambda + \sigma_k^2), \tag{54}$$

where $\prime$ denotes $k \neq j$. By equating the constant terms in (52) and (54) we deduce

$$\left(\prod_{j=1}^{n-1} \sigma_j^2\right) \sigma_n^2 - \sum_{j=1}^{n-1} b_j^2 \prod_{k=1}^{n-1\,\prime} \sigma_k^2 = \prod_{j=1}^{n} \omega_j^2, \tag{55}$$

while by equating the coefficients of λ^{2n-1} we obtain

$$\sum_{j=1}^{n-1} c_j + c_n = \sum_{j=1}^{n} d_j. \tag{56}$$

We comment on these equations. The quantities $(\sigma_k^2)_1^{n-1}$ and $(\omega_j^2)_1^n$ may be computed, via (51), (53) respectively, from the data. Once real $(b_j)_1^{n-1}$ are known, equation (55) gives σ_n^2, and ensures that $\sigma_n^2 > 0$. Equation (56) gives c_n in terms of $(d_j)_1^n$, and $(c_j)_1^{n-1}$, which again are given, via (53), (51) respectively, in terms of the data.

For convenience we will call the complete system S, and the system constrained so that $u_n = 0$, S_L. We note that if S is undamped then S_L is undamped. For the stated contraints on the eigenvalues $\lambda_j, \lambda'_j, \mu_j, \mu'_j$ mean that all c_j, d_j are non-negative. If S is undamped, i.e., $(d_j)_1^n = 0$, then equation (56) implies $(c_j)_1^n = 0$. If $g_n = 0$, then the converse is true: if S_L is undamped then S is undamped. For if S_L is undamped, then $(c_j)_1^{n-1} = 0$, so that, from equation (49), $g_j + G_j = 0$, $j = 1, 2, \ldots, n-1$. Therefore, since $g_j \geq 0$, $G_j \geq 0$, we must have $(g_j, G_j)_1^{n-1} = 0$, and thus $g = 0$ and $c_n = 0$. Now $d_j \geq 0$ and equation (56) implies $(d_j)_1^n = 0$; S is undamped.

Now we must find the $(2n-2)$ quantities $(a_j, b_j)_1^{n-1}$. We obtain them by equating (52) and (54) for the s pairs $(\mu_j, \mu'_j)_1^s$ and for the $n-1-s$ pairs $(\mu_j, \overline{\mu}_j)_{s+1}^{n-1}$. First we proceed formally. We have

$$(a_j\mu_j + b_j)^2 = \{R(\mu_j)\}^2, \quad (a_j\mu'_j + b_j)^2 = \{R(\mu'_j)\}^2, \quad j = 1, \ldots, s \tag{57}$$

$$(a_j\mu_j + b_j)^2 = \{R(\mu_j)\}^2, \quad (a_j\overline{\mu}_j + b_j)^2 = \{R(\overline{\mu}_j)\}^2, \quad j = s+1, \ldots, n-1 \tag{58}$$

where

$$\{R(\mu)\}^2 = \frac{-\prod_{k=1}^{n}(\mu^2 + d_k\mu + \omega_k^2)}{\prod_{k=1}^{n-1}{}'(\mu^2 + c_k\mu + \sigma_k^2)} = \frac{N(\mu)}{D(\mu)}. \tag{59}$$

Note that $R(\mu)$ is evaluated for one of $\mu_j, \mu'_j, \overline{\mu}_j$; $'$ denotes $k \neq j$.

The two pairs (57) and (58) are fundamentally different. First, consider (57); μ_j, μ'_j are real and negative so that, for a real solution a_j, b_j with $a_j \geq 0, b_j > 0$, $\{R(\mu_j)\}^2$ and $\{R(\mu'_j)\}^2$ must be finite and non-negative. In this case, i.e., $1 \leq j \leq s$, the numerator and denominator of $\{R(\mu_j)\}^2$ are

$$N(\mu_j) = -\prod_{k=1}^{r}(\mu_j - \lambda_k)(\mu_j - \lambda'_k) \cdot \prod_{k=r+1}^{n}(\mu_j - \lambda_k)(\mu_j - \overline{\lambda}_k) = A(\mu_j) \cdot B(\mu_j) \tag{60}$$

$$D(\mu_j) = -\prod_{k=1}^{s}{}'(\mu_j - \mu_k)(\mu_j - \mu'_k) \cdot \prod_{k=s+1}^{n}(\mu_j - \mu_k)(\mu_j - \overline{\mu}_k) = C(\mu_j) \cdot E(\mu_j) \tag{61}$$

Similarly

$$\{R(\mu'_j)\}^2 = N(\mu'_j)/D(\mu'_j) = A(\mu'_j)B(\mu'_j)/\{C(\mu'_j)E(\mu'_j)\}. \tag{62}$$

If $\{R(\mu_j)\}^2$ and $\{R(\mu'_j)\}^2$ are to be finite, then $D(\mu_j), D(\mu'_j)$ must be non-zero, and thus the μ_j, μ'_j must be distinct. We order them so that

$$\mu_1 < \mu'_1 < \mu_2 < \mu'_2 < \ldots < \mu'_s < 0 \tag{63}$$

and we suppose that the $(\lambda_j, \lambda'_j)_1^r$ may also be so ordered, i.e.,

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \ldots < \lambda'_r < 0 \tag{64}$$

The quantities $B(\mu_j), B(\mu'_j), E(\mu_j), E(\mu'_j)$, being squares of norms of non-zero complex quantities, are positive. Thus the conditions for $\{R(\mu_j)\}^2$ and $\{R(\mu'_j)\}^2$ to be non-negative, are that

$$A(\mu_j)C(\mu_j) \geq 0, \quad A(\mu'_j)C(\mu'_j) \geq 0. \tag{65}$$

We note that (63) implies $C(\mu_j) \neq 0, C(\mu'_j) \neq 0$. When written in full, (65) gives

$$-\prod_{k=1}^{r}(\mu_j - \lambda_k)(\mu_j - \lambda'_k) \cdot \prod_{k=1}^{s'}(\mu_j - \mu_k)(\mu_j - \mu'_k) \geq 0, \quad j = 1, 2, \ldots, s, \tag{66}$$

$$-\prod_{k=1}^{r}(\mu'_j - \lambda_k)(\mu'_j - \lambda'_k) \cdot \prod_{k=1}^{s'}(\mu'_j - \mu_k)(\mu'_j - \mu'_k) \geq 0, \quad j = 1, 2, \ldots, s. \tag{67}$$

It may be verified that the necessary and sufficient conditions for these inequalities to hold are that each μ_j lie between a λ_k, λ'_k pair, and each μ'_j also lie between a λ_l, λ'_l pair; these pairs may be the same or different, both for one pair μ_j, μ'_j and for two or more pairs μ_j, μ'_j, as shown in Figure 5. In particular, we note that if S_L has an overdamped pair μ_j, μ'_j, then S must have *at least one* overdamped pair also. When the inequalities (66), (67) are satisfied, then $\{R(\mu_j)\}^2, \{R(\mu'_j)\}^2$ are

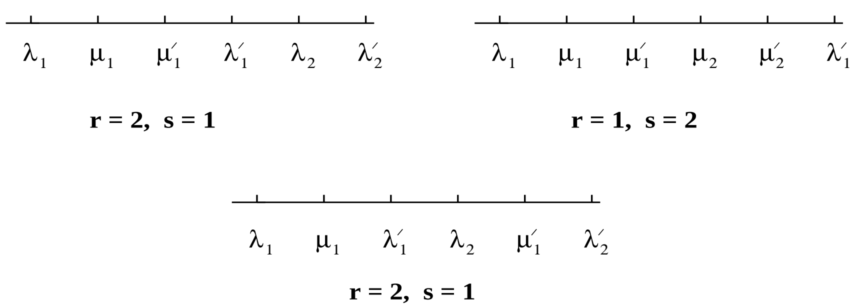

Figure 5: Three samples of possible μ, λ configurations for overdamped S_L eigenvalues.

non-negative, and equations (57) may be replaced by

$$a_j\mu_j + b_j = \pm R(\mu_j), \quad a_j\mu'_j + b_j = \pm R(\mu'_j), \tag{68}$$

where $R(\mu_j) = R$ and $R(\mu'_j) = R'$ denote the non-negative square roots. In general, there will be four solutions for a_j, b_j; we may write these as

$$a_j^{(1)}, b_j^{(1)}; a_j^{(2)}, b_j^{(2)}; -a_j^{(1)}, -b_j^{(1)}; -a_j^{(2)}, -b_j^{(2)}.$$

Suppressing the index j, we have

$$a^{(1)} = \frac{R' + R}{\mu' - \mu}, \quad b^{(1)} = \frac{-\mu' R - \mu R'}{\mu' - \mu}, \tag{69}$$

$$a^{(2)} = \frac{R' - R}{\mu' - \mu}, \quad b^{(2)} = \frac{\mu' R - \mu R'}{\mu' - \mu}. \tag{70}$$

Since R, R' are both non-negative and $\mu < \mu' < 0$, the first solution has $a^{(1)} \geq 0,\ b^{(1)} \geq 0$; the inequalities will be strict unless $R = 0 = R'$. We note that one of R, R' can be zero, and will be zero when a μ_j or μ'_j $(j = 1, \ldots, s)$ equals a λ_k or a λ'_k $(k = 1, 2, \ldots, r)$. We conclude that the necessary and sufficient condition for there to be a solution $a_j^{(1)}, b_j^{(1)}$ with $a_j^{(1)} \geq 0, b_j^{(1)} > 0$ is that μ_j, μ'_j lie between a λ_k, λ'_k pair, and one of μ_j, μ'_j lies strictly between a λ_k, λ'_k pair. We note that if $a_j^{(1)} \geq 0, b_j^{(1)} > 0$, then $a_j^{(1)} > 0$. We note that the second solution may or may not be positive.

Now consider equation (58). Since μ_j is complex we may take the square roots of the complex quantities $\{R(\mu_j)\}^2$ and $\{R(\overline{\mu}_j)\}^2$ and get

$$a_j \mu_j + b_j = \pm R(\mu_j), \qquad a_j \overline{\mu}_j + b_j = \pm \overline{R(\mu_j)}. \tag{71}$$

Now, for consistency we must take the same signs in the two equations. If we write

$$\{R(\mu_j)\}^2 = |R_j|^2 e^{2i\theta_j}, \qquad 0 \leq \theta_j < \pi, \tag{72}$$

and take

$$\mu_j = \rho_j e^{i\alpha_j}, \qquad \frac{\pi}{2} < \alpha_j < \pi \tag{73}$$

then with $R(\mu_j) = |R_j| e^{i\theta_j}$ and the positive signs in (71) we find

$$a_j = \frac{R_j \ sin \ \theta_j}{\rho_j \ sin \ \alpha_j}, \qquad b_j = \frac{R_j \ sin(\alpha_j - \theta_j)}{sin \ \alpha_j}. \tag{74}$$

If the system is to be a connected system then we must have $a_j \geq 0$ and $b_j > 0$. This means that we must have

$$0 \leq \theta_j < \alpha_j, \qquad j = s+1, \ldots, n-1. \tag{75}$$

We have now found the conditions for the data to correspond to a real quadratic pencil. We must now investigate when and whether we can find real positive masses, stiffnesses and damping factors from the quadratic pencil. To do this, we must invert the equations (44), (49) and (50). Put $m_n = 1, m_i = u_i^2$, then

$$\left.\begin{array}{ll} g_i = u_i a_i, & k_i = u_i b_i, \\ G_i = u_i^2 c_i - u_i a_i, & K_i = u_i^2 \sigma_i^2 - u_i b_i, \end{array}\right\} i = 1, 2, \ldots, n-1 \tag{76}$$

$$g_n = c_n - \sum_{j=1}^{n-1} u_j a_j, \tag{77}$$

$$k_n = \sigma_n^2 - \sum_{j=1}^{n-1} u_j b_j. \tag{78}$$

The problem of finding positive $u_i, \ i = 1, 2, \ldots, n-1$ is a linear programming problem. The given spectral data will be realizable if the following problem has a solution:

$$u_i > 0, \quad u_i \geq a_i/c_i, \quad u_i > b_i/\sigma_i^2 \tag{79}$$

$$c_n - \sum_{j=1}^{n-1} a_j u_j \geq 0, \quad \sigma_n^2 - \sum_{j=1}^{n-1} b_j u_j > 0. \tag{80}$$

First, we consider the two lower bounds: a_j/c_j and b_j/σ_j^2. Suppose the jth mode is underdamped, then

$$\frac{a_j}{c_j} - \frac{b_j}{\sigma_j^2} = \frac{a_j \sigma_j^2 - b_j c_j}{c_j \sigma_j^2}, \tag{81}$$

and

$$a_j \sigma_j^2 - b_j c_j = \frac{R_j}{sin \ \alpha_j} \{ sin \ \theta_j - 2 \ cos \ \alpha_j \ sin(\alpha_j - \theta_j) \}. \tag{82}$$

The inequalities $0 \leq \theta_j < \pi$ imply $sin \ \theta_j \geq 0$, and the inequalities $\frac{\pi}{2} < \alpha_j < \pi, \ \alpha_j > \theta_j$ imply $-cos \ \alpha_j \ sin(\alpha_j - \theta_j) > 0$. Thus $a_j \sigma_j^2 - b_j c_j > 0$. If the mode is overdamped, then

$$a_j \sigma_j^2 - b_j c_j = \frac{\mu_j' R_j + \mu_j^2 R_j'}{(\mu_j^{2\prime} - \mu_j^2) \mu_j \mu_j'} > 0. \tag{83}$$

Thus in all cases

$$\frac{a_j}{c_j} > \frac{b_j}{\sigma_j^2}. \tag{84}$$

We must therefore take $u_j = (a_j/c_j) + x_j,\ x_j \geq 0,\ j = 1, \ldots, n-1$ so that the inequalities (80) become

$$c_n - \sum_{j=1}^{n} \frac{a_j^2}{c_j} - \sum_{j=1}^{n-1} a_j x_j \geq 0 \tag{85}$$

$$\sigma_n^2 - \sum_{j=1}^{n} \frac{a_j b_j}{c_j} - \sum_{j=1}^{n-1} b_j x_j > 0 \tag{86}$$

The conditions for the existence of a non-negative solution $x_1, x_2, \ldots, x_{n-1}$ are thus

$$c_n - \sum_{j=1}^{n-1} \frac{a_j^2}{c_j} \geq 0, \quad \sigma_n^2 - \sum_{j=1}^{n-1} \frac{a_j b_j}{c_j} > 0 \tag{87}$$

The first condition states that **C** is ps-d. The inequality (84) shows that the second condition is somewhat stronger than the condition

$$\sigma_n^2 - \sum_{j=1}^{n-1} \frac{b_j^2}{\sigma_j^2} > 0 \tag{88}$$

which states that **K** is positive definite.

6 *RECAPITULATION AND CONCLUSIONS*

The procedure described in Section 5 is somewhat involved, and it is difficult to see it as a whole. We therefore list the principal steps in the analysis, to see what we have established, and what remains to be established.

The data are the eigenvalues:

$$\lambda_j, \lambda'_j, j = 1, 2, \ldots, r \quad ; \quad \lambda_j, \overline{\lambda}_j, j = r+1, \ldots, n.$$
$$\mu_j, \mu'_j, j = 1, 2, \ldots, s \quad ; \quad \mu_j, \overline{\mu}_j, j = s+1, \ldots, n-1.$$

The data yield the quantities c_j, σ_j^2 through equation (51);

$$c_j = -\mu_j - \mu'_j, \quad \sigma_j^2 = \mu_j \mu'_j, \quad j = 1, 2, \ldots, s \tag{89}$$

$$c_j = -\mu_j - \overline{\mu}_j, \quad \sigma_j^2 = \mu_j \overline{\mu}_j, \quad j = s+1, \ldots, n-1 \tag{90}$$

and simularly give the quantities d_j, ω_j^2 through equation (53):

$$d_j = -\lambda_j - \lambda_j', \quad \omega_j^2 = \lambda_j \lambda_j', \quad j = 1, 2, \ldots, r \tag{91}$$

$$d_j = -\lambda_j - \overline{\lambda}_j, \quad \omega_j^2 = \lambda_j \lambda_j', \quad j = r+1, \ldots, n. \tag{92}$$

Now $(c_j)_1^{n-1}$ and $(d_j)_1^n$ yield c_n through equation (56):

$$c_n = \sum_{j=1}^{n} d_j - \sum_{j=1}^{n-1} c_j. \tag{93}$$

Since $c_n > 0$, this equation yields the first necessary condition on the eigenvalue data:

$$-\sum_{j=1}^{r}(\lambda_j + \lambda_j') - \sum_{j=r+1}^{n}(\lambda_j + \overline{\lambda}_j) + \sum_{j=1}^{s}(\mu_j + \mu_j') + \sum_{j=s+1}^{n-1}(\mu_j + \overline{\mu}_j) > 0. \tag{94}$$

Now we pass to the next stage; the evaluation of the a_j and b_j. From the eigenvalue data we form the quantities $R(\mu_j),\ R(\mu_j'),\ j = 1, 2, \ldots, s$. Then equations (69) give

$$a_j = \frac{R(\mu_j') + R(\mu_j)}{\mu_j' - \mu_j}, \quad b_j = \frac{-\mu_j' R(\mu_j) - \mu_j R(\mu_j')}{\mu_j' - \mu_j}, j = 1, 2, \ldots, s. \tag{95}$$

These will be finite and positive if the overdamped eigenvalues satisfy the inequalities (63) and (64), namely

$$\mu_1 < \mu_1' < \mu_2 < \mu_2' < \ldots \quad < \mu_s' < 0 \tag{96}$$

$$\lambda_1 < \lambda_1' < \lambda_2 < \lambda_2' < \ldots \quad < \lambda_r' < 0 \tag{97}$$

$$\tag{98}$$

and, in addition, each μ_j, μ_j' lies between a λ_k, λ_k' pair, with one of each μ_j, μ_j' lying strictly between such a pair.

The corresponding formulae, for the a_j, b_j relating to underdamped modes, are given by (72)-(74):

$$\{R(\mu_j)\}^2 = R_j^2 \ exp(2i\theta_j), \quad 0 \le \theta_j < \pi \tag{99}$$

$$\mu_j = \rho_j \ exp(i\alpha_j) \quad \frac{\pi}{2} < \alpha_j < \pi \tag{100}$$

$$a_j = \frac{R_j \; sin \; \theta_j}{\rho_j \; sin \; \alpha_j}, \quad b_j = \frac{R_j \; sin(\alpha_j - \theta_j)}{sin \; \alpha_j} \tag{101}$$

Now the simple interlacing conditions satisfied by the overdamped eigenvalues are replaced by inequalities which involve all the eigenvalues $(\lambda_j)_1^n$ and individual eigenvalues μ_j which appear in $R(\mu_j)$; the inequalities are (75), namely

$$0 \leq \theta_j < \alpha_j, \qquad j = s+1, \ldots, n-1. \tag{102}$$

These conditions, and the corresponding interlacing conditions for the overdamped eigenvalues, ensure that the a_j, b_j are positive. Now equation (55) yields the last quantity, σ_n^2, appearing in the reduced matrix $\mathbf{Q}(\lambda)$:

$$\sigma_n^2 = \left\{ \prod_{j=1}^{n} w_j^2 + \sum_{j=1}^{n-1} b_j^2 \prod_{k=1}^{n-1\prime} \sigma_k^2 \right\} / \prod_{j=1}^{n-1} \sigma_j^2. \tag{103}$$

At this point we are assured that the data corresponds to a real quadratic pencil, and we enter the last stage: the determination of the masses $m_i = u_i^2$. The analysis given in equations (76)-(87) shows that the inequalities (87) must hold, and if they do then we may take $x_j = 0$, i.e.,

$$m_n = 1, \quad m_j = u_j^2, \quad u_j = a_j/c_j \tag{104}$$

Now for $j = 1, 2, \ldots, n-1$, we have

$$g_j = \frac{a_j^2}{c_j}, \quad k_j = \frac{a_j b_j}{c_j}, \quad G_j = 0, \quad K_j = \frac{a_j}{c_j}\left(\frac{a_j \sigma_j^2}{c_j} - b_j\right) \tag{105}$$

while

$$g_n = c_n - \sum_{j=1}^{n-1} a_j^2/c_j, \quad k_n = \sigma_n^2 - \sum_{j=1}^{n-1} a_j b_j/c_j. \tag{106}$$

If the inequality (85) is strict, then we find an infinite family of other solutions with positive x_j corresponding to positive G_j, where now the parameters are given by the more general equations (76)-(78) with $a_j = (a_j/c_j) + x_j, \; x_j > 0, \; j = 1, 2, \ldots, n-1$.

7 *CONCLUSIONS*

For a linear undamped vibrating system there is a strong and simple statement which can be made regarding the effect of a constraint: *the constrained eigenvalues interlace the unconstrained eigenvalues.* This paper has attempted to elucidate what happens when the system is subject to (viscous) damping. Now the eigenvalues of both the original and constrained systems are complex. We have studied a particularly simple system with n degrees of freedom, with $(n-1)$ masses in parallel. We found that there are interlacing conditions on *overdamped* (negative real) eigenvalue pairs, but that the conditions on the complex pairs of unconstrained and constrained eigenvalues are not simple interlacing inequalities – they cannot be, because the eigenvalues lie in the complex plane, and points in the plane cannot be ordered – but are more complicated inequalities involving all the eigenvalues at once.

Part II will discuss a theoretical-experimental study of the simplest case studied here, $n = 2$.

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