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ON SOME EIGENVECTOR-EIGENVALUE RELATIONS*

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Abstract. This paper generalizes the well-known identity which relates the last components of the eigenvectors of a symmetric matrix $\mathbf{A}$ to the eigenvalues of $\mathbf{A}$ and of the matrix $\mathbf{A}_{n-1}$, obtained by deleting the last row and column of $\mathbf{A}$. The generalizations relate to matrices and to Sturm–Liouville equations.

Key words. eigenvalue, eigenvector, matrix, Sturm–Liouville equation, Green's function

AMS subject classifications. 15A18, 15A24, 34B25, 34B27

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1. Introduction. We use the term *Jacobi matrix* to denote a real symmetric tridiagonal matrix with positive off-diagonal terms. It is well known that a Jacobi matrix $\mathbf{A}$ may be uniquely constructed from the eigenvalues $\{\lambda_i\}_1^n$ and $\{\mu_i\}_1^{n-1}$ of $\mathbf{A}$ and of its leading principal minor $\mathbf{A}_{n-1}$, respectively. The first step in the reconstruction procedure is to use the given eigenvalues, which must interlace so that

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n \tag{1.1}$$

to yield the last elements, $x_n^{(i)}$, of the normalized eigenvectors $\mathbf{x^{(i)}}$ of $\mathbf{A}$. We use the fact that $\{\mu_j\}_1^{n-1}$ are the zeros of

$$\sum_{i=1}^{n} \frac{[x_n^{(k)}]^2}{\lambda_i - \lambda} = 0, \tag{1.2}$$

so that

$$\sum_{k=1}^{n} \frac{[x_n^{(k)}]^2}{\lambda_k - \lambda} = c\frac{\prod_{j=1}^{n-1}(\mu_j - \lambda)}{\prod_{j=1}^{n}(\lambda_j - \lambda)}. \tag{1.3}$$

The identity $\sum_{k=1}^{n}[x_n^{(k)}]^2 = 1$ implies $c = 1$, and thus

$$[x_n^{(k)}]^2 = \frac{\prod_{j=1}^{n-1}(\mu_j - \lambda_k)}{\prod_{j=1,j\neq k}^{n}(\lambda_j - \lambda_k)}. \tag{1.4}$$

This relation, which has been known for many years (see, say, [7]), is an example of what we term an *eigenvector-eigenvalue relation*.

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Many other such relations are known; all involve the eigenvalues of $\mathbf{A}$ and of some modification of $\mathbf{A}$. As another example, if $\{\lambda_i^*\}_1^n$ are the eigenvalues of the Jacobi matrix $\mathbf{A}^*$, obtained by replacing the last diagonal element a_n by a_n^*, then [6]

$$1 + (a_n^* - a_n) \sum_{k=1}^{n} \frac{[x_n^{(k)}]^2}{\lambda_k - \lambda} = \prod_{j=1}^{n} \left(\frac{\lambda_j^* - \lambda}{\lambda_j - \lambda} \right), \tag{1.5}$$

from which we deduce

$$(a_n^* - a_n)[x_n^{(k)}]^2 = \frac{\prod_{j=1}^{n}(\lambda_j^* - \lambda_k)}{\prod_{j=1, j\neq k}^{n}(\lambda_j - \lambda_k)}. \tag{1.6}$$

By comparing the traces of $\mathbf{A}$ and $\mathbf{A}^*$ we obtain

$$(a_n^* - a_n) = \sum_{k=1}^{n}(\lambda_k^* - \lambda_k). \tag{1.7}$$

Equations (1.6), (1.7) provide an eigenvector-eigenvalue relation; it is a particular case of a relation referring to the eigenvalues to $\mathbf{A}$ and some rank-one modification of A; see [10].

Equations (1.4), (1.5), and (1.6) relate to a Jacobi matrix; there are analogous results for continuous systems governed by Sturm–Liouville equations. We give an example, the analogue of (1.5), not the most general result.

Let $\{\lambda_i\}_0^\infty$, $\{\lambda_i^*\}_0^\infty$ be the eigenvalues of the nonuniform string equation

$$y''(x) + \lambda\rho(x)y(x) = 0, \tag{1.8}$$

subject to two sets of end conditions

$$y'(0) - hy(0) = 0 = y'(l) + Hy(l), \tag{1.9}$$

$$y'(0) - hy(0) = 0 = y'(l) + H^*y(l), \tag{1.10}$$

differing only at the right end, and let $\rho(x)$ be continuous; then the end values $y_m(l)$ of the normalized eigenfunctions of (1.8), subject to the condition (1.9), satisfy the equation

$$1 + (H^* - H) \sum_{m=0}^{\infty} \frac{[y_m(l)]^2}{\lambda_m - \lambda} = c\frac{\prod_{m=0}^{\infty}(1 - \frac{\lambda}{\lambda_m^*})}{\prod_{m=0}^{\infty}(1 - \frac{\lambda}{\lambda_m})}. \tag{1.11}$$

Since λ_m, $\lambda_m^* = O(m^2)$ for large m, both infinite products converge. Again we find

$$(H^* - H)[y_n(l)]^2 = c\lambda_n \prod_{m=0}^{\infty} \left(1 - \frac{\lambda_n}{\lambda_m^*}\right) \Bigg/ \prod_{m=0}^{\infty}{}' \left(1 - \frac{\lambda_n}{\lambda_m}\right), \tag{1.12}$$

where $c\prod_{m=0}^{\infty}(\frac{\lambda_m}{\lambda_m^*}) = 1$, and the prime means $m \neq n$. This example appears in [8] and is rederived in [6, p. 180].

The purpose of this paper is to explore some generalizations of the relations we have described. They will refer to discrete and continuous systems. They will involve squares of eigenvector/eigenfunction values at interior points and products of such values at two different points.

2. The generalized eigenvalue problem. Let $\mathbf{A}$, $\mathbf{B}$ be symmetric matrices of order n, with $\mathbf{B}$ positive definite, and let $\{\lambda_i\}_1^n$, $\{\mathbf{x}^{(i)}\}_1^n$ be the eigenpairs of

$$(\mathbf{A} - \lambda\mathbf{B})\mathbf{x} = \mathbf{0}; \tag{2.1}$$

then $\mathbf{A}$, $\mathbf{B}$ may be simultaneously diagonalized so that

$$\mathbf{X}^T\mathbf{A}\mathbf{X} = \wedge, \qquad \mathbf{X^T}\mathbf{B}\mathbf{X} = \mathbf{I}, \tag{2.2}$$

where $\wedge = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\mathbf{X} = [\mathbf{x^{(1)}}, \mathbf{x^{(2)}}, \ldots, \mathbf{x^{(n)}}]$, and

$$\mathbf{A} - \lambda\mathbf{B} = \mathbf{X}^{-T}(\wedge - \lambda\mathbf{I})\mathbf{X}^{-1}. \tag{2.3}$$

Provided that $\lambda \neq \lambda_i$, $i = 1, 2, \ldots, n$, we may invert this to give

$$(\mathbf{A} - \lambda\mathbf{B})^{-1} = \mathbf{X}(\wedge - \lambda\mathbf{I})^{-1}\mathbf{X}^T. \tag{2.4}$$

This gives the following lemma.

LEMMA 2.1. *Provided the eigenvalue λ_k is simple, then*

$$x_i^{(k)}x_j^{(k)} = \lim_{\lambda\to\lambda_k}(\lambda_k - \lambda)\mathbf{e_i^T}(\mathbf{A} - \lambda\mathbf{B})^{-1}\mathbf{e_j} \tag{2.5}$$

where $\mathbf{e}_i$ denotes the ith column of $\mathbf{I}$.

Proof. Let $\alpha_{ij}(\lambda)$ be the i, j element of $(\mathbf{A} - \lambda\mathbf{B})^{-1}$, so that

$$\alpha_{ij}(\lambda) = \mathbf{e}_i^T(\mathbf{A} - \lambda\mathbf{B})^{-1}\mathbf{e}_j. \tag{2.6}$$

Then (2.4) gives

$$\alpha_{ij}(\lambda) = \sum_{k=1}^{n} \frac{x_i^{(k)}x_j^{(k)}}{\lambda_k - \lambda} \tag{2.7}$$

so that, provided the eigenvalue λ_k is simple,

$$x_i^{(k)}x_j^{(k)} \quad = \lim\nolimits_{\lambda\to\lambda_k}(\lambda_k - \lambda)\alpha_{ij}(\lambda). \qquad \square \tag{2.8}$$

We now apply this result with some special choices of $\mathbf{A}$ and $\mathbf{B}$. We introduce some notation. Let $\mathbf{A}_k(\mathbf{A}_k^R)$, $1 \leq k \leq n$, denote the leading (trailing) principal submatrix of order k of the square matrix $\mathbf{A}$. We let $P_k(\lambda)$,$Q_k(\lambda)$ denote, respectively, the kth order leading and trailing principal minors of $\mathbf{A} - \lambda\mathbf{B}$, and $(\mu_i)_1^{n-1}$ denote the zeros of $P_{n-1}(\lambda)$.

THEOREM 2.1.

$$[x_n^{(k)}]^2 = \frac{|\mathbf{B}_{n-1}|\prod_{j=1}^{n-1}(\mu_j - \lambda_k)}{|\mathbf{B}_n|\prod_{j=1}^{n\prime}(\lambda_j - \lambda_k)}, \tag{2.9}$$

where $\prime$ denotes $j \neq k$.

Proof. Consider the equation

$$(\mathbf{A} - \lambda\mathbf{B})\mathbf{y} = \mathbf{e}_n. \tag{2.10}$$

If $\lambda \neq \lambda_i$ $(i = 1, 2, \ldots, n)$, then

$$y_n = \mathbf{e}_n^T(\mathbf{A} - \lambda\mathbf{B})^{-1}\mathbf{e}_n = \alpha_{nn}(\lambda). \tag{2.11}$$

But Cramer's rule applied to (2.10) gives

$$y_n = \frac{|\mathbf{A}_{n-1} - \lambda \mathbf{B}_{n-1}|}{|\mathbf{A}_n - \lambda \mathbf{B}_n|} = \frac{P_{n-1}(\lambda)}{P_n(\lambda)}. \tag{2.12}$$

But $\{\mu_i\}_1^{n-1}$ are the zeros of $P_{n-1}(\lambda)$ and $(\lambda_i)_1^n$ are the zeros of $P_n(\lambda)$, so that

$$\alpha_{nn}(\lambda) = y_n = \frac{|\mathbf{B}_{n-1}| \prod_{j=1}^{n-1}(\mu_j - \lambda)}{|\mathbf{B}_n| \prod_{j=1}^{n}(\lambda_j - \lambda)}, \tag{2.13}$$

and Lemma 2.1, with $i = j = n$, gives

$$[x_n^{(k)}]^2 = \frac{|\mathbf{B}_{n-1}| \prod_{j=1}^{n-1}(\mu_j - \lambda_k)}{|\mathbf{B}_n| \prod_{j=1}^{n\prime}(\lambda_j - \lambda_k)}. \qquad \square \tag{2.14}$$

Theorem 2.1 generalizes (1.4). It holds provided that λ_k is simple. We recall that the eigenvalues of a Jacobi matrix (to which (1.4) applies) are always simple, and the eigenvalues λ_i, μ_i appearing in (1.4) always *strictly* interlace according to (1.1), so that (1.4) never breaks down and $[x_n^{(k)}]^2$ is always positive; $x_n^{(k)}$ is never zero. For the generalized eigenvalue problem (2.1), the eigenvalues need not be simple, and the eigenvalues λ_i, μ_i satisfy only

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \cdots \le \mu_{n-1} \le \lambda_n. \tag{2.15}$$

If λ_k is simple, so that $\lambda_{k-1} < \lambda_k < \lambda_{k+1}$, then (2.9) holds for that value of k; $x_n^{(k)}$ will be zero if $\mu_{k-1} = \lambda_k$ or $\mu_{k+1} = \lambda_k$. If λ_k is not simple, then we have the following corollary.

COROLLARY 2.1. *Suppose λ_k has multiplicity s, so that*

$$\lambda_{k-1} < \lambda_k = \mu_k = \lambda_{k+1} = \cdots = \mu_{k+s-2} = \lambda_{k+s-1} \le \mu_{k+s-1} < \lambda_{k+s}.$$

Then

$$[x_n^{(k)}]^2 = \frac{|B_{n-1}| \prod_{j=1}^{n-1}{}'(\mu_j - \lambda_k)}{|B_n| \prod_{j=1}^{n}{}''(\lambda_j - \lambda_k)}, \tag{2.16}$$

where $'$ means $j = 1, 2, \ldots, k-1$, $k+s-1, \ldots, n-1$ and $''$ means $j = 1, 2, \ldots, k-1$, $k+s, \ldots, n$.

Proof. The numerator and denominator of (2.14) will have a common factor $(\lambda_k - \lambda)^{s-1}$ which can be cancelled. $\square$

The results in Theorem 2.1 and its corollary are obtained from Lemma 2.1 for $i = j = n$. There are analogous expressions for $[x_1^{(k)}]^2$, but there are, in general, no simple extensions to $[x_m^{(k)}]^2$ where $1 < m < n$. To obtain simple extensions we must restrict the forms of $\mathbf{A}$ and $\mathbf{B}$ and suppose that they are both tridiagonal.

We obtain the following lemma.

LEMMA 2.2. *Suppose $\mathbf{A}$ and $\mathbf{B}$ are tridiagonal matrices with codiagonals $(b_1, b_2, \ldots, b_{n-1})$ and $(d_1, d_2, \ldots, d_{n-1})$, respectively, and suppose $1 \le i \le j \le n$ and $\lambda \ne \lambda_k$, $k = 1, 2, \ldots, n$. Then*

$$\alpha_{ij}(\lambda) = (-1)^{i+j} \frac{P_{i-1}(\lambda) \prod_{k=i}^{j-1}(b_k - \lambda d_k) Q_{n-j}(\lambda)}{P_n(\lambda)}. \tag{2.17}$$

Proof. Consider the equation

$$(\mathbf{A} - \lambda\mathbf{B})y = \mathbf{e}_j. \tag{2.18}$$

If $\lambda \neq \lambda_k$, $k = 1, 2, \ldots, n$, then

$$y_i = \mathbf{e}_i^T(\mathbf{A} - \lambda\mathbf{B})^{-1}\mathbf{e}_j = \alpha_{ij}(\lambda). \tag{2.19}$$

But Cramer's rule applied to (2.18) shows that

$$\alpha_{ij}(\lambda) = y_i = \frac{\hat{p}_{ij}(\lambda)}{p_n(\lambda)}, \tag{2.20}$$

where $\hat{p}_{ij}(\lambda)$ is the determinant of the matrix obtained by replacing the ith column of $\mathbf{A} - \lambda\mathbf{B}$ by $\mathbf{e}_j$. Thus

$$\hat{p}_{ij}(\lambda) = \begin{array}{c} \begin{matrix} & & & i & & & \end{matrix} \\ \begin{vmatrix} a_1 - \lambda c_1 & b_1 - \lambda d_1 & & 0 & & & \\ b_1 - \lambda d_1 & a_2 - \lambda c_2 & \cdots & 0 & & & \\ \cdots & \cdots & \cdots & 1 & & & \\ & \cdots & & 0 & \cdots & a_{n-1} - \lambda c_{n-1} & b_{n-1} - \lambda d_{n-1} \\ & & & 0 & \cdots & b_{n-1} - \lambda d_{n-1} & a_n - \lambda c_n \end{vmatrix} \end{array} \; j. \tag{2.21}$$

Assuming that $i \leq j$, and expanding $\hat{p}_{ij}(\lambda)$ along its ith column, we obtain the stated result. ∎

Note that the numerator involves a leading minor, a product, and a trailing minor.

Let us consider the interpretation of (2.18). The original equation may be interpreted as the equation governing the free vibration of a mechanical system with stiffness and inertia matrices $\mathbf{A}$, $\mathbf{B}$, respectively. The eigenvalues λ_i are the squares, ω_i^2, of the natural frequencies; and the $\mathbf{x}_i$, the eigenvectors, are the mode shapes. The term $\alpha_{ij}(\lambda)$ is the influence function linking coordinates i and j: it gives the solution y_i for a unit in the jth place on the right-hand side of (2.14). We can think of this as the *displacement* at i due to a unit *load* at j; thus $\alpha_{ij}(\lambda)$ is the *receptance* [3] between points i and j. Equation (2.18) thus states that $\alpha_{ij}(\lambda)$ is zero when $P_{i-1}(\lambda) = 0$, $Q_{n-j}(\lambda) = 0$, or the product is zero. But the zeros of $P_{i-1}(\lambda)$ are simply the eigenvalues $(\lambda_k^L)_1^{i-1}$ of the equation

$$(\mathbf{A}_{i-1} - \lambda\mathbf{B}_{i-1})\mathbf{x} = \mathbf{0}, \tag{2.22}$$

while the zeros of $Q_{n-j}(\lambda)$ are the eigenvalues $(\lambda_k^R)_1^{n-j}$ of the equation

$$(\mathbf{A}_{n-j}^R - \lambda\mathbf{B}_{n-j}^R)\mathbf{x} = \mathbf{0}. \tag{2.23}$$

(Remember that $A_k(A_k^R)$ is the leading (trailing) principal submatrix of order k.) In the important physical case in which $b_k < 0$ and $d_k \geq 0$, the product in (2.18) has no positive zero; we examine this application in section 5.

We now have the following theorem.

THEOREM 2.2. *Suppose* $\mathbf{B}$ *is diagonal,* $1 \leq i \leq j \leq n$, *and* $1 \leq k \leq n$. *Then*

$$\frac{x_i^{(k)} x_j^{(k)}}{\alpha_{ij}(0)} = \frac{\lambda_k \prod_{m=1}^{i-1}(1 - \frac{\lambda_k}{\lambda_m^L}) \prod_{m=1}^{n-j}(1 - \frac{\lambda_k}{\lambda_m^R})}{\prod_{m=1}^{n}{}'(1 - \frac{\lambda_k}{\lambda_m})}. \tag{2.24}$$

Proof. When $\mathbf{B}$ is diagonal, $d_r = 0$, so that (2.17) reduces to

$$\alpha_{ij} = c\frac{P_{i-1}(\lambda)Q_{n-j}(\lambda)}{P_n(\lambda)}, \qquad c = (-1)^{i+j}\prod_{r=i}^{j-1} b_r. \tag{2.25}$$

On normalizing α_{ij} by its value for $\lambda = 0$, and by factorizing the various terms, we find

$$\Phi_{ij}(\lambda) \equiv \frac{\alpha_{ij}(\lambda)}{\alpha_{ij}(0)} = \frac{\prod_{m=1}^{i-1}(1-\frac{\lambda}{\lambda_m^L})\prod_{m=1}^{n-j}(1-\frac{\lambda}{\lambda_m^R})}{\prod_{m=1}^{n}(1-\frac{\lambda}{\lambda_m})}. \tag{2.26}$$

Using this in (2.6), we find the required result. ∎

We note that, when $\mathbf{B}$ is diagonal, the $(\lambda_m)_1^n$ are distinct, so that the denominator in (2.24) cannot be zero. The numerator can be zero, since one of each of the sets $(\lambda_m^L)_1^{i-1}$ and $(\lambda_m^R)_1^{n-j}$ can be equal to λ_k. Theorem 2.1, Corollary 2.1, and Theorem 2.2 provide generalization of the known result (1.4).

We now explore some continuous analogues of these results.

3. Sturm–Liouville systems.

Let L denote the Sturm–Liouville operator given by

$$Ly(x) = -(p(x)y'(x))' + q(x)y(x), \tag{3.1}$$

and consider the system

$$Ly(x) = \lambda\rho(x)y(x), \tag{3.2}$$

subject to the end conditions

$$p(0)y'(0) - hy(0) = 0 = p(l)y'(l) + Hy(l). \tag{3.3}$$

Such a system can model the free vibration of a rod or string fastened at its ends by springs of stiffness h, H, respectively; h or H is zero at a free end, infinite at a fixed end.

We are not concerned here with the most general regularity conditions satisfied by p, q, and ρ. We assume that $p(x)$, $p'(x)$, $q(x)$, and $\rho(x)$ are continuous in $(0, l)$. References to the general theory may be found in [9], [1], and [2].

The *Green*'s *function* $G(x, s, \lambda)$ for the system satisfies, as a function of x

(a) the equation (3.2), except at s,

(b) the end conditions (3.3),

(c) the jump condition

$$[p(x)y'(x)]_{x=s-}^{x=s+} = -1. \tag{3.4}$$

It is well known [4, Chapter 5] that if λ is not an eigenvalue of (3.2), (3.3), then $G(x, s, \lambda)$ may be constructed as follows. Let $\phi(x)$, $\psi(x)$ be solutions of (3.2) satisfying

$$p(0)\phi'(0) - h\phi(0) = 0 = p(l)\psi'(l) + H\psi(l), \tag{3.5}$$

respectively; then

$$p(x)\{\phi(x)\psi'(x) - \phi'(x)\psi(x)\} = \text{constant}. \tag{3.6}$$

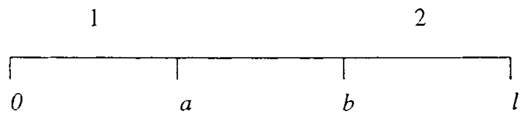

FIG. 1.

This constant is zero if λ is an eigenvalue of (3.2), (3.3) and nonzero otherwise; in the latter case we can choose the constant to be -1, and then

$$G(x,s,\lambda) = \begin{cases} \phi(x)\psi(s), & 0 \le x \le s, \\ \phi(s)\psi(x), & s \le x \le l. \end{cases} \tag{3.7}$$

The functions ϕ, ψ are functions of x and λ; if $\lambda = 0$ is not an eigenvalue of (3.2), (3.3), i.e., if h, H are not both zero, then

$$G(x,s,0) = \begin{cases} \phi_0(x)\psi_0(s), & 0 \le x \le s, \\ \phi_0(s)\psi_0(x), & s \le x \le l, \end{cases} \tag{3.8}$$

where $\phi_0(x)$, $\psi_0(x)$ denote $\phi(x)$, $\psi(x)$, respectively, for $\lambda = 0$.

The Green's function $G(x,s,\lambda)$ is the analogue of the receptance $\alpha_{ij}(\lambda)$ for matrix systems; like $\alpha_{ij}(\lambda)$, it has poles and zeros. In section 2 we explicitly showed $\alpha_{ij}(\lambda)$ as a product of two polynomials $P_{i-1}(\lambda)$, $Q_{n-j}(\lambda)$ divided by $P_n(\lambda)$. The polynomials $P_{i-1}(\lambda)$, $Q_{n-j}(\lambda)$ related to the parts of the system, respectively, to the left of i and to the right of j, and their zeros were the eigenvalues of these parts, as shown in (2.22), (2.23). We will now show how the Green's function $G(a,b,\lambda)$, with $a \le b$, may be expressed in an analogous way as a product of two quantities referring, respectively, to the parts of the system to the left of a, and to the right of b, divided by a third quantity relating to the whole interval $(0,l)$.

Suppose $a \le b$, and define the normalized Green's function

$$\Phi(a,b,\lambda) = \frac{G(a,b,\lambda)}{G(a,b,0)} = \frac{\phi(a)\psi(b)}{\phi_0(a)\psi_0(b)}. \tag{3.9}$$

Let $\{\lambda_n^L\}_1^\infty$, $\{\lambda_n^R\}_1^\infty$ be the eigenvalues of the subsystems S_1, S_2 governed by (3.2) with the end conditions

$$1. \quad p(0)y'(0) - hy(0) = 0, \qquad y(a) = 0, \tag{3.10}$$

$$2. \quad y(b) = 0, \qquad p(l)y'(l) + Hy(l) = 0, \tag{3.11}$$

and let $\{\lambda_n\}_1^\infty$ be the eigenvalues of (3.1)–(3.3). We prove the following theorem.

THEOREM 3.1.

$$\Phi(a,b,\lambda) = \frac{\prod_{n=1}^\infty (1 - \frac{\lambda}{\lambda_n^L}) \prod_{n=1}^\infty (1 - \frac{\lambda}{\lambda_n^R})}{\prod_{n=1}^\infty (1 - \frac{\lambda}{\lambda_n})}. \tag{3.12}$$

Proof. Fig. 1 shows the system divided into three parts.

To find the Green's function of system 1, we need function $\phi_1(x)$, $\psi_1(x)$ satisfying, respectively, the left and right end conditions of system 1 and the jump condition; we may take

$$\phi_1(x) = \phi(x), \qquad \psi_1(x) = \psi(x) - \frac{\psi(a)}{\phi(a)}\phi(x). \tag{3.13}$$

We now introduce the function

$$H_1(\lambda) = \lim_{\varepsilon \to 0} \frac{G_1(\varepsilon, a-\varepsilon, 0)}{G_1(\varepsilon, a-\varepsilon, \lambda)} \tag{3.14}$$

$$= \lim_{\varepsilon \to 0} \frac{\phi_{1,0}(\varepsilon)}{\phi_1(\varepsilon)} \cdot \frac{\psi_{1,0}(a-\varepsilon)}{\psi_1(a-\varepsilon)}, \tag{3.15}$$

where $\phi_{1,0}(x)$, $\psi_{1,0}(x)$ denote, respectively, the values of $\phi_1(x)$, $\psi_1(x)$ for $\lambda = 0$. Since $\psi_1(a) = 0 = \psi_{1,0}(a)$, we have

$$\lim_{\varepsilon \to 0} \frac{\psi_{1,0}(a-\varepsilon)}{\psi_1(a-\varepsilon)} = \frac{\psi'_{1,0}(a)}{\psi'_1(a)} = \frac{\phi(a)}{\phi_0(a)} \tag{3.16}$$

because of (3.6). The limit of the first quotient in (3.14) is

$$\lim_{\varepsilon \to 0} \frac{\phi_{1,0}(\varepsilon)}{\phi_1(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\phi_0(\varepsilon)}{\phi_1(\varepsilon)} = \frac{\phi_0(0)}{\phi(0)} \tag{3.17}$$

if h is finite, and $\phi'_0(0)/\phi'(0)$ if h is infinite. Thus

$$H_1(\lambda) = \begin{cases} \frac{\phi_0(0)}{\phi(0)} \cdot \frac{\phi(a)}{\phi_0(a)} & \text{if } h \text{ is finite,} \\ \frac{\phi'_0(0)}{\phi'(0)} \cdot \frac{\phi(a)}{\phi_0(a)} & \text{if } h \text{ is infinite.} \end{cases} \tag{3.18}$$

We may now consider system 2 similarly. We take

$$\phi_2(x) = \phi(x) - \frac{\phi(b)}{\psi(b)}\psi(x), \qquad \psi_2(x) = \psi(x), \tag{3.19}$$

define

$$H_2(\lambda) = \lim_{\varepsilon \to 0} \frac{G_2(b+\varepsilon, l-\varepsilon, 0)}{G_2(b+\varepsilon, l-\varepsilon, \lambda)}, \tag{3.20}$$

and find

$$H_2(\lambda) = \begin{cases} \frac{\psi(b)}{\psi_0(b)} \cdot \frac{\psi_0(l)}{\psi(l)} & \text{if } H \text{ is finite,} \\ \frac{\psi(b)}{\psi_0(b)} \cdot \frac{\psi'_0(l)}{\psi'(l)} & \text{if } H \text{ is infinite.} \end{cases} \tag{3.21}$$

Finally we take the whole system, with end conditions (3.3), and define

$$H(\lambda) = \lim_{\varepsilon \to 0} \frac{G(\varepsilon, l-\varepsilon, 0)}{G(\varepsilon, l-\varepsilon, \lambda)} \tag{3.22}$$

and find that this has one of the values

$$\frac{\phi_0(0)\psi_0(l)}{\phi(0)\psi(l)}, \quad \frac{\phi'_0(0)\psi_0(l)}{\phi'(0)\psi(l)}, \quad \frac{\phi_0(0)\psi'_0(l)}{\phi(0)\psi'(l)}, \quad \frac{\phi'_0(0)\psi'_0(l)}{\phi'(0)\psi'(l)} \tag{3.23}$$

according to whether both, one, the other, or neither of h, H are finite.

Combining (3.18), (3.21), (3.22) we find the fundamental relation for $\Phi(a, b, \lambda)$ defined in (3.9):

$$\Phi(a, b, \lambda) = \frac{H_1(\lambda) H_2(\lambda)}{H(\lambda)}. \tag{3.24}$$

We now show that $H_1(\lambda)$, $H_2(\lambda)$, and $H(\lambda)$ may, like $P_{i-1}(\lambda)$, $Q_{n-j}(\lambda)$ and $P_n(\lambda)$ in (2.21), be expressed as product of factors relating to the subsystems 1, 2, and the whole.

First consider the case when h is finite. It is known (see, for example, [11]) that, treated as a function of λ, the quantity $\phi(a)/\phi(0)$ is an entire function of λ, of order 1/2. By Hadamard's factorization theorem it may therefore be expressed as a product of its factors; i.e.,

$$\frac{\phi(a)}{\phi(0)} = c\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{a_n}\right). \tag{3.25}$$

But the zeros of $\phi(a)$ are precisely the eigenvalues of system 1 (i.e., λ_n^L) so that $a_n = \lambda_n^L$. It is known [11] that $\lambda_n^L = O(n^2)$ for large n, so that the product (3.24) converges. The constant c is $\phi_0(a)/\phi_0(0)$. Thus

$$H_1(\lambda) = \frac{\phi(a)}{\phi(0)}\cdot\frac{\phi_0(0)}{\phi_0(a)} = \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n^L}\right). \tag{3.26}$$

If h is infinite, $H_1(\lambda)$ is defined by the second line of (3.18), but the final result for $H_1(\lambda)$ still holds.

We may deduce, in a similar way, that

$$H_2(\lambda) = \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n^R}\right). \tag{3.27}$$

To investigate $H(\lambda)$, we note that the jump condition (3.6) and second of the end conditions (3.5) give

$$\psi(l)\{p(l)\phi'(l)+H\phi(l)\} = 1 = \psi_0(l)\{p(l)\phi_0'(l)+H\phi_0(l)\} \tag{3.28}$$

so that when h, H are finite, and not both zero,

$$H(\lambda) = \frac{p(l)\phi'(l)+H\phi(l)}{\phi(0)}\cdot\frac{\phi_0(0)}{p(l)\phi_0'(l)+H\phi_0(l)}. \tag{3.29}$$

Again, $H(\lambda)$ is an entire function of order 1/2, whose zeros are those λ for which there is a solution $\phi(x)$ satisfying *both* end conditions (3.3), i.e., $\lambda = \lambda_n$. Thus

$$H(\lambda) = \prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right). \tag{3.30}$$

This result still holds when one or both of h, H are infinite.

Equations (3.24), (3.26), (3.27), (3.29) now yield the stated result (3.12). □

COROLLARY 3.1. *Let* $\{u_r(x)\}$ *be the eigenfunctions of* (3.1)–(3.3)*; then*

$$\frac{u_r(a)u_r(b)}{\sum_{r=1}^{\infty}\frac{u_r(a)u_r(b)}{\lambda_r}} = \frac{\lambda_r\prod_{n=1}^{\infty}(1-\frac{\lambda_r}{\lambda_n^L})\prod_{n=1}^{\infty}(1-\frac{\lambda_r}{\lambda_n^R})}{\prod_{n=1}^{\infty}{}'(1-\frac{\lambda_r}{\lambda_n})}. \tag{3.31}$$

Proof. The usual modal expansion of $G(a,b,\lambda)$, namely,

$$G(a,b,\lambda) = \sum_{r=1}^{\infty}\frac{u_r(a)u_r(b)}{\lambda_r-\lambda}, \tag{3.32}$$

when substituted in (3.9), gives

$$\Phi(a,b,\lambda)=\frac{\sum_{r=1}^{\infty}\frac{u_r(a)u_r(b)}{\lambda_r-\lambda}}{\sum_{r=1}^{\infty}\frac{u_r(a)u_r(b)}{\lambda_r}}, \tag{3.33}$$

which, when combined with Theorem 3.1, yields the required result. □

We note that

$$\sum_{r=1}^{\infty}\frac{u_r(a)u_r(b)}{\lambda_r}=\phi_0(a)\psi_0(b), \tag{3.34}$$

and $\phi_0(a),\psi_0(b)$ may be expressed explicitly in terms of $p(x)$, $q(x)$, h, H.

4. Conclusion. Corollary 3.1 provides the desired continuous analogue of Theorem 2.2. The analysis shows that the continuous analogues of the leading and trailing minors are the functions $H_1(\lambda)$, $H_2(\lambda)$, defined in (3.14), (3.20). We note that $H_1(\lambda)$, $H_2(\lambda)$ are not just reciprocals of the Green's functions, as one might expect at first thought, but limiting values of the normalized reciprocals of the Green's functions between the points near the ends of the intervals (0, a) and (b, l).

Various other eigenvector-eigenvalue relations may be obtained from Theorem 2.2 by taking $i=j$, $i=1$, or $j=n$ or in Corollary 3.1 by taking $a=b$, $a=0$, or $b=l$.

5. An application. Consider the mass spring system shown in Fig. 2. This is the customary model of an in-line axially vibrating system; by replacing the masses and axial springs by polar inertias and torsional springs, respectively, we obtain the analogous torsionally vibrating system. We show that the analysis may be used to find those frequencies of oscillation at which a particular displacement vanishes. These frequencies are the vibration absorber frequencies which play an important role in vibration isolation. We may tune, for example, the working frequencies of an unbalanced motor, mounted on a shaft, to eliminate the steady state vibrations at a certain position where sensitive mechanical or electrical equipment is located, such as isolation from vibration and noise-induced vibration usually required in aircraft structures. (For other applications, see [5, Sections 3.2, 3.3, 4.2].)

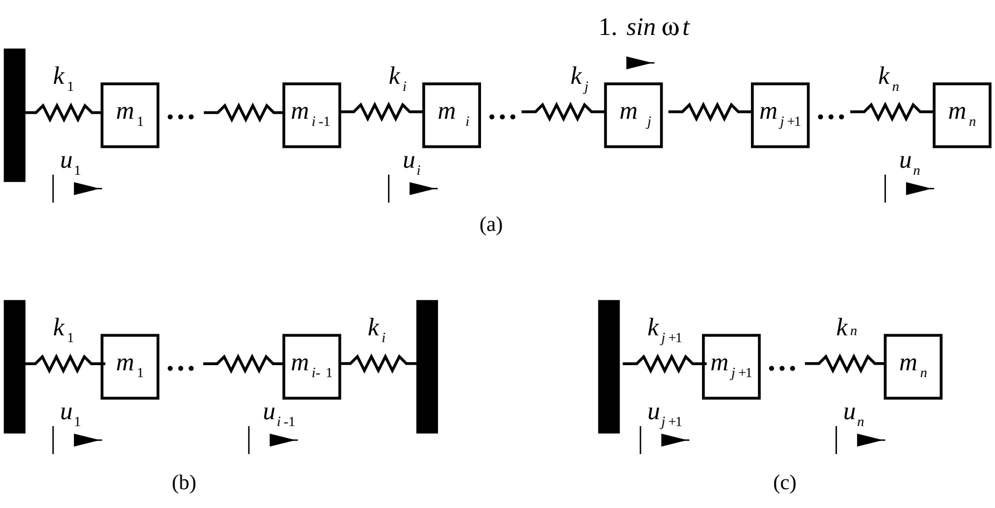

FIG. 2. *A spring-mass system* (a) *has left and right end parts shown in* (b) *and* (c), *respectively.*

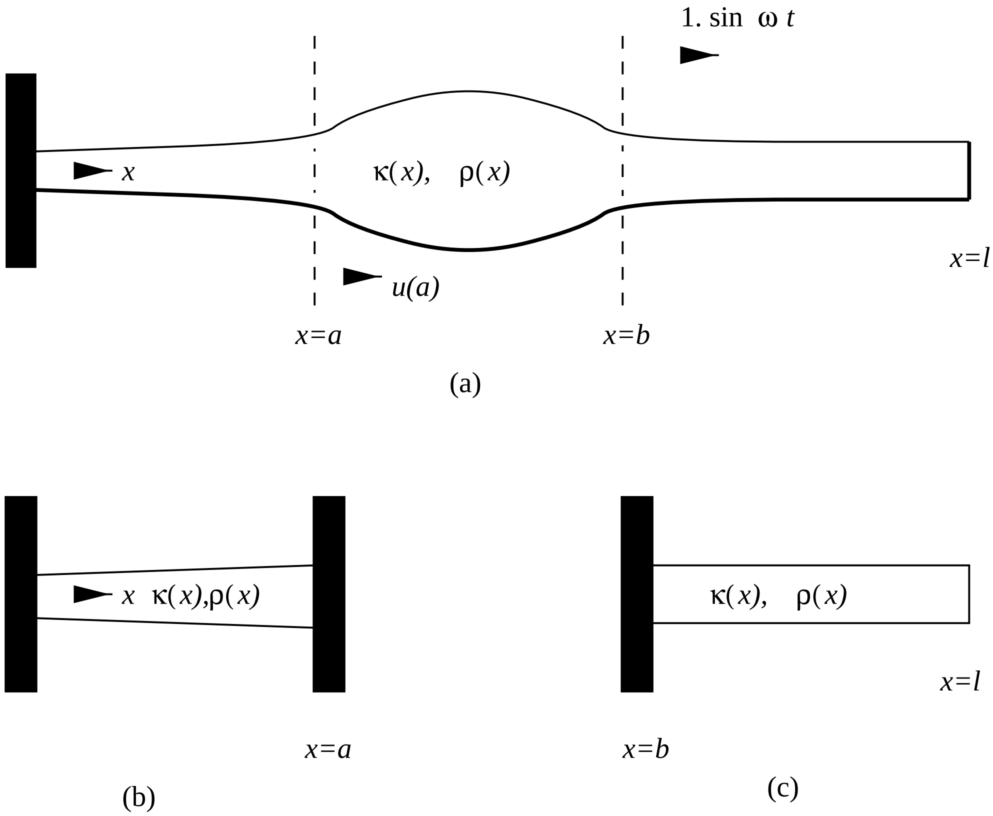

FIG. 3. *The rod* (a) *has left and right end parts* (b) *and* (c).

The receptance α_{ij} in (2.6) gives the response at mass i due to a unit load with frequency ω ($\omega^2 = \lambda$) at mass j. Without loss of generality we can consider just the case $i \le j$. The expression (2.22) for α_{ij} shows that u_i will be zero if $\lambda = \lambda_m^L$ for some $m = 1, 2, \ldots, i-1$ or $\lambda = \lambda_m^L$ for some $m = 1, 2, \ldots, n-j$. Moreover, by applying superposition, we see that if loads are applied at masses $i+1, \ldots, n$ with the same frequency ω ($\omega^2 = \lambda$), then u_i will be zero if $\lambda = \lambda_m^L$ for some $m = 1, 2, \ldots, i-1$.

The continuous analogue of the discrete system of Fig. 2 is the axially vibrating rod shown in Fig. 3a. Now we apply Theorem 3.1. If a load with frequency ω ($\omega^2 = \lambda$) is applied at $x = b$, then the displacement at $x = a$ will be zero if $\lambda = \lambda_n^L$ or $\lambda = \lambda_n^R$ for some n. By superposition, we see that if loads are applied to the part $a < x \le l$ with the same frequency ω, then $u(a)$ will be zero if $\lambda = \lambda_n^L$ for some n.

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