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On transforming three-dimensional inverse scattering problems to one dimension*

M Chamberland†§ and G M L Gladwell‡

† Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

‡ Solid Mechanics Division, Department of Civil Engineering, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

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Abstract. It is well known that three-dimensional inverse scattering problems are much more difficult to solve than one-dimensional problems. For this reason, some authors have considered cases where one may transform three-dimensional problems to one dimension, in essence looking for three-dimensional problems which allow a one-dimensional wave-splitting. We show here that such a reduction can be carried out only in three specific cases.

Consider the three-dimensional acoustic wave equation

$$\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = \frac{1}{\rho c^2}\frac{\partial^2 p}{\partial t^2} \tag{1}$$

coupled with the boundary conditions

$$\begin{aligned} p(x,t) &= p_0(x,t) && \text{on} \quad S_0 \\ \frac{\partial p}{\partial n}(x,t) &= \sigma_0(x,t) && \text{on} \quad S_0 \end{aligned}$$

where S_0 is a two-dimensional smooth closed surface and the n-derivative is the normal derivative to the surface S_0. Here we assume p is a C^2 function and that ρ and c are continuous. We impose the quiescent condition, namely $p \equiv 0$ for $t < 0$. We also add the extra reasonable assumption that the function $p(u_1, \cdot, \cdot, t)$ has compact support. In general, we assume $c = c(\mathbf{x})$ and $\rho = \rho(\mathbf{x})$. If the functions c, ρ and p_0 are given, a forward problem would be to solve for σ_0. If p_0 and σ_0 are given, an inverse problem would be to solve for c and ρ.

When the wave speed c is constant, the problem may be transformed into the Schrödinger equation. The inverse problem in this setting has received considerable attention (see for example Newton [9] and Rose *et al* [10]). If ρ is assumed to be constant, (1) reduces to the classical wave equation

$$\nabla^2 p(x,t) = \frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2}p(x,t)\,. \tag{2}$$

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§ E-mail address: mchamberland@violet.uwaterloo.ca

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Weston [13–16] has given much attention to this equation by applying wave-splitting methods which identify 'upward' and 'downward' directions of the wave motion. This was done in an attempt to extend the layer-stripping methods used succesfully in solving one-dimensional inverse problems (see for example Bruckstein *et al* [1]), but the application to inverse problems has not been thoroughly investigated.

Several authors consider problems with a three-dimensional equation with undetermined coefficients which depend only on one dimension. The planar case (c and ρ are functions of z) was considered by Weston [13] and Coen [2]. Taking advantage of the geometry, Coen used the Radon transform to transform the problem into a one-dimensional one. We note that while the one-dimensional inverse problems yield only the ratio of ρ and c, Coen generates a family of one-dimensional problems which allowed him to determine these two functions individually. Coen [3] also considered the spherical case, and Kreider [7] and Weston [14] studied the circular cylindrical case. The papers of Weston and Kreider obtain the Ricatti-type integro-differential equation used by Corones *et al* [4].

In this paper we wish to find all the cases when a three-dimensional problem may be reduced to a one-dimensional one. Weston [14] (who worked with the classical wave equation) considered the situation where one has an orthogonal curvilinear coordinate system (u_1, u_2, u_3) (see [11] for an elaboration on curvilinear coordinates) and $c = c(u_1)$. We denote the corresponding scale factors by h_1, h_2, h_3. Weston claims a reduction to one dimension is realized when $\frac{\partial}{\partial u_1}\ln(h_2 h_3)$ and h_1 are functions of u_1 only. The planar case considered in [13] fits the stated criteria. Two non-planar examples considered were the cases of spherical and circular cylindrical geometry. We show that the conditions Weston imposes to obtain a one-dimensional problem are not only sufficient, but they are also necessary to reduce such a three-dimensional problem to one dimension. Secondly, only the planar, spherical and circular cylindrical geometries satisfy these conditions.

A system will be considered reducible to one dimension if it may be written

$$\frac{\partial}{\partial u_1}\left(a(u_1)\frac{\partial q}{\partial u_1}\right) = b(u_1)\frac{\partial^2 q}{\partial t^2} \tag{3}$$

for then it may be replaced by the pair of equations

$$a(u_1)\frac{\partial q}{\partial u_1} = -\frac{\partial r}{\partial t} \qquad \frac{\partial r}{\partial u_1} = -b(u_1)\frac{\partial q}{\partial t}. \tag{4}$$

With the change to the travel-time coordinate

$$\xi = \int^{u_1}\sqrt{\frac{b}{a}}\,\mathrm{d}u_1$$

we may reduce (4) to the standard form

$$\frac{\partial q}{\partial \xi} = -Z(\xi)\frac{\partial r}{\partial t} \qquad \frac{\partial q}{\partial t} = -Z(\xi)\frac{\partial r}{\partial \xi} \tag{5}$$

where the impedance $Z(\xi)$ is given by

$$Z(\xi) = \frac{1}{\sqrt{ab}}.$$

We require that the transformation from (1) to (3) be independent of the data p_0, σ_0.

Weston's conditions are equivalent to

$$h_1 = h_1(u_1) \qquad h_2 h_3 = f(u_1) g(u_2, u_3) . \tag{6}$$

Define

$$q(u_1, t) = \int_S g(u_2, u_3) p(u_1, u_2, u_3, t)\, \mathrm{d}u_2 \mathrm{d}u_3 = \int_S \frac{p}{f} \mathrm{d}A$$

where $\mathrm{d}A = h_2 h_3 \mathrm{d}u_2 \mathrm{d}u_3$ is an element of the surface S, $u_1 =$ constant. We will show that, under Weston's conditions (6), $q(u_1, t)$ satisfies a one-dimensional equation of the form (3). To do this we need a generalization of Leibniz's Rule for differentiating under the integral sign, due to Flanders [6], namely

$$\frac{\partial}{\partial u_1} \int w \, \mathrm{d}A = \int \frac{1}{h_2 h_3} \frac{\partial}{\partial u_1} (h_2 h_3 w) \, \mathrm{d}A \tag{7}$$

where again the integrals are taken over $u_1 =$ constant.

In orthogonal curvilinear coodinates

$$\nabla^2 p = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial p}{\partial u_1} \right) + \nabla_\perp^2 p \tag{8}$$

where the transverse Laplacian is

$$\nabla_\perp^2 p = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial p}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial p}{\partial u_3} \right) \right] .$$

Thus

$$\begin{aligned} \int_S \nabla_\perp^2 p \, \mathrm{d}A &= \int_S \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} \left(\frac{h_3}{h_2} \frac{\partial p}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_2}{h_3} \frac{\partial p}{\partial u_3} \right) \right] \mathrm{d}A \\ &= \int_S \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (h_3 F_3) - \frac{\partial}{\partial u_3} (h_2 F_2) \right] \mathrm{d}A = \int_S \boldsymbol{e}_1 \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) \, \mathrm{d}A \end{aligned}$$

where

$$\boldsymbol{F} = (0, F_2, F_3) \qquad F_2 = -\frac{1}{h_3} \frac{\partial p}{\partial u_3} \qquad F_3 = \frac{1}{h_2} \frac{\partial p}{\partial u_2}$$

and $\boldsymbol{e}_1$ is the unit length normal to the surface S. If S is bounded, then it must be diffeomorphic to a sphere. Let C be a curve which maps into a great circle on the sphere. This curve separates the surface into two parts S_1 and S_2. Thus Stokes' theorem applied to S_1 and S_2 separately gives

$$\int_S \nabla_\perp^2 p \, \mathrm{d}A = \boldsymbol{e}_1 \cdot \int_C \boldsymbol{F} \cdot \mathrm{d}s - \boldsymbol{e}_1 \cdot \int_C \boldsymbol{F} \cdot \mathrm{d}s = 0$$

because C is traversed one way as the boundary of S_1, the other way as the boundary of S_2.

If the surface is unbounded then, since $p(u_1, \cdot, \cdot, t)$ has compact support, there is a Jordan curve C on the surface S such that C surrounds this support, and on it $F_2 = 0 = F_3$. Again the integral of the transverse Laplacian over S is zero.

Thus in any case equation (8) gives

$$h_1 \int_S \nabla^2 p \, \mathrm{d}A = \int \frac{1}{h_2 h_3} \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial p}{\partial u_1} \right) \mathrm{d}A \, .$$

On account of Flanders' Equation (7), this may be written

$$\begin{aligned} h_1 \int_S \nabla^2 p \, \mathrm{d}A &= \frac{\partial}{\partial u_1} \left(\frac{1}{h_1} \int_S \frac{\partial p}{\partial u_1} \mathrm{d}A \right) = \frac{\partial}{\partial u_1} \left(\frac{f}{h_1} \int_S \frac{1}{h_2 h_3} \frac{\partial (gp)}{\partial u_1} \mathrm{d}A \right) \\ &= \frac{\partial}{\partial u_1} \left(\frac{f}{h_1} \int_S \frac{1}{h_2 h_3} \frac{\partial}{\partial u_1} \left(h_2 h_3 \frac{p}{f} \right) \mathrm{d}A \right) = \frac{\partial}{\partial u_1} \left(\frac{f}{h_1} \frac{\partial}{\partial u_1} \int_S \frac{p}{f} \mathrm{d}A \right) \\ &= \frac{\partial}{\partial u_1} \left(\frac{f}{h_1} \frac{\partial q}{\partial u_1} \right) . \end{aligned}$$

Integrating (2) over S we obtain

$$\frac{1}{c^2(u_1)} \frac{\partial^2 q}{\partial t^2} = \frac{1}{h_1 f} \frac{\partial}{\partial u_1} \left(\frac{f}{h_1} \frac{\partial q}{\partial u_1} \right) .$$

which has the required form (3).

To prove that equations (6) are necessary, assume a solution of the form

$$q(u_1, t) := \int_S g(u_1, u_2, u_3) p(u_1, u_2, u_3, t) \, \mathrm{d}u_2 \, \mathrm{d}u_3$$

may be reduced to the form (3). Note the u_1-dependence on the g term. Equation (3) implies

$$\begin{aligned} 0 &= \int_S \left\{ \frac{\partial}{\partial u_1} \left(a \frac{\partial}{\partial u_1} (gp) \right) - bg \frac{\partial^2 p}{\partial t^2} \right\} \mathrm{d}u_2 \, \mathrm{d}u_3 \\ &= \int_S \left\{ \frac{\partial}{\partial u_1} \left(a \frac{\partial}{\partial u_1} (gp) \right) - bgc^2 \nabla^2 p \right\} \mathrm{d}u_2 \, \mathrm{d}u_3 \, . \end{aligned} \tag{9}$$

Putting

$$r = \frac{bc^2 g}{h_2 h_3}$$

we examine the expression

$$\int_S bc^2 g \nabla^2 p \, \mathrm{d}u_2 \, \mathrm{d}u_3 = \int_S r \nabla^2 p \, \mathrm{d}A \, .$$

We express the Laplacian as

$$\nabla^2 p = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial p}{\partial u_1} \right) + e_1 \cdot \nabla \times F + \frac{1}{h_1 h_2 h_3} \left[\frac{\partial h_1}{\partial u_2} \frac{h_3}{h_2} \frac{\partial p}{\partial u_2} + \frac{\partial h_1}{\partial u_3} \frac{h_2}{h_3} \frac{\partial p}{\partial u_3} \right] . \tag{10}$$

Since

$$\nabla \times (r\boldsymbol{F}) = r(\nabla \times \boldsymbol{F}) + \nabla r \times \boldsymbol{F}$$

we use Stokes' theorem as before to obtain

$$\begin{aligned}\int_S r\nabla^2 p\, \mathrm{d}A = &\int_S \frac{r}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial p}{\partial u_1}\right)\mathrm{d}A \\ &-\int_S \frac{1}{h_2 h_3}\left[\frac{h_3}{h_2}\frac{\partial p}{\partial u_2}\frac{\partial r}{\partial u_2} + \frac{h_2}{h_3}\frac{\partial p}{\partial u_3}\frac{\partial r}{\partial u_3}\right]\mathrm{d}A \\ &+\int_S \frac{r}{h_1 h_2 h_3}\left[\frac{\partial h_1}{\partial u_2}\frac{h_3}{h_2}\frac{\partial p}{\partial u_2} + \frac{\partial h_1}{\partial u_3}\frac{h_2}{h_3}\frac{\partial p}{\partial u_3}\right]\mathrm{d}A\,.\end{aligned}$$

Collecting the coefficients of $\partial p/\partial u_2$ and $\partial p/\partial u_3$ and setting them equal to zero yields

$$\frac{\partial r}{\partial u_2} - \frac{r}{h_1}\frac{\partial h_1}{\partial u_2} = 0 \qquad \frac{\partial r}{\partial u_3} - \frac{r}{h_1}\frac{\partial h_1}{\partial u_3} = 0\,.$$

This gives

$$\frac{\partial}{\partial u_2}\left(\frac{r}{h_1}\right) = 0 = \frac{\partial}{\partial u_3}\left(\frac{r}{h_1}\right)$$

which implies

$$\frac{r}{h_1} = f(u_1).$$

Continuing the calculation in (9), and letting the subscript 1 denote differentiation with respect u_1, we have

$$\begin{aligned}0 = &\int_S \left\{\frac{\partial}{\partial u_1}\left(a\frac{\partial}{\partial u_1}(gp)\right) - f(u_1)\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial p}{\partial u_1}\right)\right\}\mathrm{d}u_2\,\mathrm{d}u_3 \\ &= \int_S \{a g_{11} + a_1 g_1\}\, p\,\mathrm{d}u_2\,\mathrm{d}u_3 + \int_S\Bigg\{2a g_1 + a_1 g \\ &\quad - f\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\right)\Bigg\}p_1\,\mathrm{d}u_2\,\mathrm{d}u_3 + \int_S\left\{ag - f(u_1)\frac{h_2 h_3}{h_1}\right\}p_{11}\,\mathrm{d}u_2\,\mathrm{d}u_3\,.\end{aligned}$$

Using the independence of the data, we have that the three coefficients must each be zero:

$$a g_{11} + a_1 g_1 = 0 \tag{11}$$

$$2a g_1 + a_1 g - f\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\right) = 0 \tag{12}$$

$$ag - f\frac{h_2 h_3}{h_1} = 0\,. \tag{13}$$

Equation (13) gives

$$h_1^2 = \frac{bc^2}{a}$$

thus $h_1 = h_1(u_1)$. We then rearrange (12) as

$$\begin{aligned} 0 &= 2ag_1 + a_1 g - f\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\right) = 2ag_1 + a_1g - f\frac{\partial}{\partial u_1}\left(\frac{ag}{f}\right) \\ &= 2ag_1 + a_1g - f\left(\frac{a_1g + ag_1}{f} - \frac{agf_1}{f^2}\right) = ag_1 + \frac{agf_1}{f} \end{aligned}$$

thus

$$\frac{\partial}{\partial u_1}(gf) = 0\,.$$

Writing $gf = d(u_2, u_3)$, we have

$$h_2h_3 = \frac{bc^2g}{fh_1} = \left(\frac{bc^2}{f^2h_1}\right)\mathrm{d}(u_2, u_3)$$

and hence h_2h_3 has the desired form. This shows that Weston's conditions are necessary for a reduction to one dimension.

We now show that only three geometries satisfy these conditions. Weston noted that the two conditions imply the mean curvature is a function of u_1 only. Indeed, the mean curvature H satisfies (see [12, pp 200–202])

$$-2H = \frac{1}{h_1h_2}\frac{\partial h_2}{\partial u_1} + \frac{1}{h_1h_3}\frac{\partial h_3}{\partial u_1} \tag{14}$$

$$= \frac{1}{h_1}\frac{\partial}{\partial u_1}\ln(h_2h_3) \tag{15}$$

$$= \frac{1}{h_1(u_1)}\frac{f'(u_1)}{f(u_1)}. \tag{16}$$

The Gaussian curvature K may be written (see [8, p 137]) as

$$K = \frac{-1}{2h_2h_3}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{h_2h_3}\frac{\partial}{\partial u_2}h_3^2\right) + \frac{\partial}{\partial u_3}\left(\frac{1}{h_2h_3}\frac{\partial}{\partial u_3}h_2^2\right)\right] \tag{17}$$

$$= \frac{-1}{h_2h_3}\left[\frac{\partial}{\partial u_2}\left(\frac{1}{h_2}\frac{\partial h_3}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{1}{h_3}\frac{\partial h_2}{\partial u_3}\right)\right]. \tag{18}$$

Since the system is orthogonal, the scale factors must satisfy certain compatibility conditions. These are embodied in the Gauss–Codazzi equations [5], which are equivalent to the *equations of Lamé*:

$$\frac{\partial^2 h_i}{\partial u_k \partial u_l} = \frac{1}{h_k}\frac{\partial h_k}{\partial u_l}\frac{\partial h_i}{\partial u_k} + \frac{1}{h_l}\frac{\partial h_l}{\partial u_k}\frac{\partial h_i}{\partial u_l} \tag{19}$$

$$\frac{\partial}{\partial u_i}\left(\frac{1}{h_i}\frac{\partial h_k}{\partial u_i}\right) + \frac{\partial}{\partial u_k}\left(\frac{1}{h_k}\frac{\partial h_i}{\partial u_k}\right) + \frac{1}{h_l^2}\frac{\partial h_i}{\partial u_l}\frac{\partial h_k}{\partial u_l} = 0 \tag{20}$$

where i, k and l take the values 1, 2 and 3 in cyclic order.

Setting $i = 2$ in (20) yields

$$\frac{\partial}{\partial u_2}\left(\frac{1}{h_2}\frac{\partial h_3}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{1}{h_3}\frac{\partial h_2}{\partial u_3}\right) + \frac{1}{h_1^2}\frac{\partial h_2}{\partial u_1}\frac{\partial h_3}{\partial u_1} = 0\,.$$

Comparing this to (18), we obtain

$$K = \frac{1}{h_1^2 h_2 h_3}\frac{\partial h_2}{\partial u_1}\frac{\partial h_3}{\partial u_1}.$$

Comparing this with (15), we note that this implies the principal curvatures k_2 and k_3 may be written as

$$k_2 := -\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial u_1} \tag{21}$$

$$k_3 := -\frac{1}{h_1 h_3}\frac{\partial h_3}{\partial u_1}\,. \tag{22}$$

Setting $i = 1$ in (20) implies

$$\frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\right) = 0\,.$$

Defining

$$H_1(u_1) := \int_0^{u_1} h_1(\xi)\,\mathrm{d}\xi$$

we obtain

$$h_2(u_1, u_2, u_3) = H_1(u_1) g_{21}(u_2, u_3) + g_{22}(u_2, u_3)\,. \tag{23}$$

Similarly, setting $i = 3$ in (20) gives

$$h_3(u_1, u_2, u_3) = H_1(u_1) g_{31}(u_2, u_3) + g_{32}(u_2, u_3)\,. \tag{24}$$

Putting these together gives

$$h_2 h_3 = H_1^2 n_1 + H_1 n_2 + n_3 \tag{25}$$

where

$$n_1 = g_{21} g_{31} \qquad n_2 = g_{21} g_{32} + g_{22} g_{31} \qquad n_3 = g_{22} g_{32}\,. \tag{26}$$

Since the expression

$$\frac{\partial}{\partial u_1}\ln(h_2 h_3)$$

is a function of u_1 only, we use (25) to obtain

$$n_i \frac{\partial n_j}{\partial u_k} = n_j \frac{\partial n_i}{\partial u_k} \tag{27}$$

where $i, j = 1, 2, 3$ and $k = 2, 3$.

If $n_1 \neq 0$, (27) may be used to give

$$\frac{n_2}{n_1} = c_2(u_1) \qquad \frac{n_3}{n_1} = c_3(u_1)\,.$$

Expanding these expressions using (26) yields

$$\frac{g_{22}}{g_{21}} = m_2(u_1) \qquad \text{and} \qquad \frac{g_{32}}{g_{31}} = m_3(u_1)\,.$$

Substituting (23) into (21) gives

$$k_2 = \frac{-g_{21}}{H_1 g_{21} + g_{22}}\,.$$

If $g_{21} = 0$ at a point, $k_2 = 0$ at that point. If $g_{21} \neq 0$, we write k_2 as

$$k_2 = \frac{-1}{H_1 + m_2}\,.$$

Thus k_2 is a function of u_1 only. Similarly, k_3 is also a function of u_1 only.

Now fix u_1 so that we consider an individual surface which has constant principal curvatures. If both curvatures are zero, then it is well known that the surface must be a plane. Suppose that both curvatures are non-zero. We then have a Weingarten surface, that is, a surface for which each of the principal curvatures is a function of the other. Suppose further that the principal curvatures are not equal. Such Weingarten surfaces have the property (see [5, p 292]) that there exists a parameter s and a function $\phi = \phi(s)$ such that the principal curvatures may be written as

$$\frac{1}{k_2} = \phi(s) \qquad \text{and} \qquad \frac{1}{k_3} = \phi(s) - s\phi'(s)\,.$$

Since our principal curvatures are constant, $\phi(s)$ is a constant function, hence the principal curvatures must be equal, which contradicts our last supposition. Thus the principal curvatures must be equal.

Up to a rigid body motion, at most one surface may have specified principal curvatures. Since the sphere, circular cylinder and plane correspond to surfaces having constant principal curvatures with exactly two, one and no principal curvatures being zero respectively, these are the only three surfaces one has for $u_1 =$ constant. Since $h_1 = h_1(u_1)$, the surfaces must be nested in a symmetric way as one varies u_1.

Finally we note that the three-dimensional acoustical equation (1) may be reduced to a one-dimensional Schrödinger equation under precisely the same conditions (equation (6)).

References

[1] Bruckstein A, Levy B C and Kailath T 1985 Differential methods in inverse scattering *SIAM J. Appl. Math.* **45** 312–35

[2] Coen S 1982 Velocity and density profiles of a layered acoustic medium from source-point data *Geophysics* **47**(6) 898–905

[3] Coen S 1983 The inverse problem of a spherically symmetric fluid earth *J. Geophys. Res.* **88** 4299–303

[4] Corones J P, Davison M E and Krueger R J 1983 Direct and inverse scattering in the time domain via invariant imbedding methods *J. Acoust. Soc. Am.* 74 1535–41

[5] Eisenhart L P 1909 *A Treatise on the Differential Geometry of Curves and Surfaces* (Ginn and Company)

[6] Flanders H 1973 Differentiation under the integral sign *Am. Math. Mo.* **43** 615–27

[7] Kreider K L 1989 A Wave splitting approach to the time dependent inverse scattering for the stratified cylinder *SIAM J. Appl. Math.* **49**(3) 932–43

[8] Millman R S and Parker G D 1977 *Elements of Differential Geometry* (Englewood Cliffs, NJ: Prentice-Hall)

[9] Newton R G 1980 Inverse scattering II. Three dimensions *J. Math. Phys.* **21** 1698–715

[10] Rose J H, Cheney M and DeFacio B 1985 Three-dimensional inverse scattering: plasma and variable velocity wave equations *J. Math. Phys.* **26** 2803–13

[11] Spiegel M R 1974 *Vector Analysis* (New York: McGraw-Hill)

[12] Spivak M 1975 *A Comprehensive Introduction to Differential Geometry III* (Boston, MA: Publish or Perish)

[13] Weston V H 1987 Factorization of the wave equation in higher dimensions *J. Math. Phys.* **28** 1061

[14] Weston V H 1988 Factorization of the wave equation in a non-planar stratified medium *J. Math. Phys.* **29** 36

[15] Weston V H 1989 Wave splitting and the reflection operator for the wave equation in $\mathbb{R}^3$ *J. Math. Phys.* **30** 2545

[16] Weston V G 1990 Invariant imbedding for the wave equation in three dimensions and the applications to the direct and inverse problems *Inverse Problems* **6** 1075