The total positivity interval

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Abstract

If \( A \in M_n \) is totally positive (TP), we determine the maximum open interval \( \mathcal{I} \) around the origin such that, if \( \mu \in \mathcal{I} \), then \( A - \mu I \) is TP. If \( A \) is TP, \( \mu \in \mathcal{I} \) and \( A - \mu I = LU \), then \( B \) defined by \( B - \mu I = UL \) is TP, and has the same total positivity interval \( \mathcal{I} \). If \( A \) is merely nonsingular and totally nonnegative (TN), or oscillatory, there need be no such interval in which \( A - \mu I \) is TN.

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1. Introduction

Totally positive, and the related terms totally nonnegative and oscillatory, are important descriptors in the characterization of matrices appearing in a variety of contexts, see Gantmakher and Krein [3], Gladwell [5].

A matrix \( A \in M_n \) is said to be totally positive (TP) (totally nonnegative (TN)) if every minor of \( A \) is positive (nonnegative). It is NTN if it is invertible and TN. It is oscillatory (O) if it is TN and a power of \( A, A^m \), is TP. If \( Z = \text{diag}(+1, -1, +1, \ldots) \) and \( ZAZ \) is O, then \( A \) is said to be sign oscillatory (SO); sign oscillatory is a particular case of sign regular.

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Cryer [2] proved that if $A$ is NTN, then it has a unique factorization $LU$ with $L$ lower triangular and having unit diagonal, $U$ upper triangular, and $B = UL$ is also NTN. We may extend Cryer’s result to matrices that are TP, O or SO. If $A$ is TP then so is $B$. If $A$ is O then it is NTN, so $B$ is NTN. A power of $A$ is TP so that $A^m = (LU)^m$ is TP, and then $B^{m+1} = (UL)^m L$ is TP, $B$ is O. Similarly if $A$ is SO, so is $B$.

For symmetric $A$, i.e., $A \in S_n$, Gladwell [6] extended Cryer’s result as follows. Let $P$ denote one of the properties TP, NTN, O or SO. If $A$ has property P, $\mu$ is not an eigenvalue of $A$, $A - \mu I = QR$ where $Q$ is orthogonal and $R$ is upper triangular with diagonals chosen to be positive, and $B$ is defined by $B - \mu I = RQ$, then $B$ also has property P. This result depends on the fact that if $A \in S_n$ and is nonsingular, then its $QR$ factorization may be effected by making two successive $LU$ factorizations. This is not true for general $A \in M_n$.

The following counterexample, with $\mu = 0$, shows that Gladwell’s result can not be extended to general $A \in M_n$:

\[
A = \begin{bmatrix} 2 & a \\ 1 & 2 \end{bmatrix}, \quad Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad R = \sqrt{5} \begin{bmatrix} 1 & (2a + 2)/5 \\ 0 & (a - 4)/5 \end{bmatrix},
\]

\[
B = \frac{1}{5} \begin{bmatrix} 12 + 2a & 4a - 1 \\ 4 - a & 2(4 - a) \end{bmatrix}.
\]

If $a = 1/5$, then $A$ is TP, but $B$ is not even TN. In Section 2 we find a restricted version of Gladwell’s result that holds for $A \in M_n$.

2. The total positivity interval

The TP-interval of a TP matrix, denoted by $\mathcal{T}_A$, is the maximum open interval around zero such that $A - \mu I$ is TP for $\mu \in \mathcal{T}_A$. We seek this interval.

Following Ando [1] we let $Q_{p, n}$ denote the set of strictly increasing sequences of $p$ integers taken from $\{1, 2, \ldots, n\}$. If $\alpha = (\alpha_1, \ldots, \alpha_p) \in Q_{p, n}$ and $\beta = (\beta_1, \ldots, \beta_q) \in Q_{q, n}$, we denote the submatrix of $A$ lying in rows indexed by $\alpha$ and columns indexed by $\beta$, by $A[\alpha|\beta]$.

When $\alpha \in Q_{p, n}$ and $\beta \in Q_{q, n}$, and $\alpha \cap \beta = \phi$, then $\alpha \cup \beta$ is rearranged increasingly to become a member of $Q_{p+q, n}$.

We use Sylvester’s identity on bordered determinants:

If $\alpha, \beta \in Q_{p, n}$ let $C = (c_{ij})$ where $c_{ij} = \det A[\alpha \cup i|\beta \cup j]$, and $\gamma, \delta \in Q_{q, n}$, then

\[
\det C[\gamma|\delta] = (\det A[\alpha|\beta])^{q-1} \det A[\alpha \cup \gamma|\beta \cup \delta].
\]

This states that if the minors of $A$ are positive, then the minors of $C$ are positive also. This matrix $C$ is bordered about the submatrix $A[\alpha|\beta]$.
Theorem 1. Suppose \( A \in M_n \) is TP. Then \( \mathcal{J}_A = (a, b) \) where \( a \) and \( b \) are defined in (5).

Proof. The corner minors of \( A \) are \( \det A[1, 2, \ldots, p|n - p + 1, \ldots, n] \) and \( \det A[n - p + 1, \ldots, n|1, 2, \ldots, p] \) for \( p = 1, 2, \ldots, n \). It is known (Gasca and Pena [4], Gladwell [6]) that if \( A \) is TN and its corner minors are strictly positive, then \( A \) is TP. We may use this result to narrow the search for the total positivity interval. Consider what happens to the minors of \( A - \mu I \) as \( \mu \) increases (decreases) from zero. Suppose if possible that one or more non-corner minors are the first to become zero, at \( \mu = \mu_0 \). At \( \mu_0 \), \( A - \mu I \) is TN but its corner minors are strictly positive; \( A \) is TP, contradicting the assumption that a minor is zero. Thus we may seek the interval in which the corner minors are positive. We examine these corner minors. Let \( m = \lfloor n/2 \rfloor \), the integral part of \( n/2 \). Consider the corner minors taken from the top right corner; these are \( \det (A - \mu I) \) for \( p = 1, 2, \ldots, m \) the corner minors are independent of \( \mu \); for \( p = m + 1, \ldots, n \) the variable \( \mu \) appears in the diagonal terms \( a_{i,i} - \mu \). We may partition the \( p \)th order submatrix, and write its determinant as follows:

\[
\Delta_p(\mu) = \begin{vmatrix}
  a_{q,q+1} & \cdots & q_{1,p} & q_{1,p+1} & \cdots & a_{1,n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{q,q+1} & \cdots & a_{q,p} & a_{q,p+1} & \cdots & a_{q,n} \\
  a_{q+1,q+1} - \mu & a_{q+1,q+1} & \cdots & a_{q+1,p+1} & \cdots & a_{q+1,n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{p,p-\mu} & a_{p,p+1} & \cdots & a_{p,n} \\
\end{vmatrix}
\]

(1)

Here \( q = n - p \). Now use Sylvester’s identity. If \( C_p = (c_{ij}) \).

\[
c_p = \det A[1, 2, \ldots, q|p + 1, \ldots, n] \\
c_{ij} = \det A[1, 2, \ldots, q, i|j, p + 1, \ldots, n], \quad i, j = q + 1, \ldots, p
\]

then, after taking account of the change of sign arising from the column interchanges, we find

\[
c_p c_{p+q-1} \Delta_p(\mu) = \det (C_p + (-)^{q-1}c_{p} \mu I_{p-q}).
\]

(2)

Sylvester’s identity shows that \( C_p \in M_{p-q} \) is TP, so that all the eigenvalues of \( C_p \) are positive. If \( q \) is odd then (2) shows that

\[
\Delta_p(\mu) > 0 \quad \text{if} \quad c_{p} \mu > -\lambda_{p, \lim, R}
\]

(3)

where \( \lambda_{p, \lim, R} \) is the least eigenvalue of \( C_p \); \( R \) denotes the fact that we are considering right-hand corner minors. If \( q \) is even then (2) shows that

\[
\Delta_p(\mu) > 0 \quad \text{if} \quad c_{p} \mu < \lambda_{p, \lim, R}.
\]

(4)

For each odd \( q \), (3) gives a lower bound for \( \mu \); for each even \( q \), (4) gives an upper bound for \( \mu \). The TP interval \( \mathcal{J}_A = (-a, b) \) is bounded by the least of these upper bounds and the greatest of the lower bounds. Thus
where $L$ denotes the eigenvalues derived from the left-hand corner minors. □

Numerical experiments indicated that there was no particular ordering among the eigenvalues $\lambda_{p,\text{min}}$ for different values of $p$. It proved to be difficult to find a TP matrix $A$ such that $A - \mu I$ loses its total positivity for a positive value of $\mu$ less than that $\lambda_1$, the lowest eigenvalue of $A$. However, for the TP matrix

$$
A = \begin{bmatrix}
1.8756 & 0.7300 & 1.2706 & 11.7002 & 8.1829 \\
1.8747 & 0.7513 & 1.3534 & 12.5589 & 8.9982 \\
1.8003 & 0.7433 & 1.3884 & 12.9948 & 9.5591 \\
1.6674 & 0.7070 & 1.3636 & 12.8930 & 9.7773 \\
1.4929 & 0.6492 & 1.2889 & 12.3143 & 9.6265
\end{bmatrix}
$$

the top right $3 \times 3$ minor of $A - \mu I$ becomes zero at $\mu = 4.1190e^{-5}$, which is less than $\lambda_1 = 0.0001$, the lowest eigenvalue of $A$; this shows that $A - \mu I$ can lose its total positivity for values of $\mu$ such that $0 < \mu < \lambda_1$.

**Corollary 2.1.** If $\mathcal{A} = (-a, b)$ is the TP interval for $A$, then $A - \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ is TP provided that $\mu_i \in \mathcal{A}$, $i = 1, 2, \ldots, n$.

**Proof.** Let

$$
\mu_r = \max_{i=1,2,\ldots,n} \mu_i, \quad \mu_s = \min_{i=1,2,\ldots,n} \mu_i.
$$

Consider the minor $\Delta_p(\mu_1, \mu_2, \ldots, \mu_n)$ obtained by replacing $\mu I_{p-q}$ by $\text{diag}(\mu_{q+1}, \ldots, \mu_p)$ in (1). If $q$ has even parity

$$
c_p^{p-q-1} \Delta_p(\mu_1, \mu_2, \ldots, \mu_n) = \det(C_p - c_p \text{diag}(\mu_{q+1}, \ldots, \mu_p))
= \det(C_p - c_p \mu_r I_{p-q} + c_p \text{diag}((\mu_r - \mu_{q+1}), \ldots, (\mu_r - \mu_p)))
\geq \det(C_p - c_p \mu_r I_{p-q}) = c_p^{p-q+1} \Delta_p(\mu_r) > 0
$$

because all the minors of $C_p - c_p \mu_r I_{p-q}$ are positive. Similarly, if $q$ has odd parity, then

$$
\Delta_p(\mu_1, \mu_2, \ldots, \mu_n) \geq \Delta_p(\mu_s) > 0. \quad \square
$$

**Theorem 2.** If $A \in M_n$ is TP and $A$ has LU factorization $A = LU$ where $L$ has unit diagonal, then $B = UL$ has the same TP interval $\mathcal{A}$ as $A : \mathcal{A} = \mathcal{A}_B$.

**Proof.** For given $p$, denote the matrix $C_p$ and scalar $c_p$ for $B$ by $D_p$, $d_p$ respectively

$$
c_p = \det A[1, 2, \ldots, q|p + 1, \ldots, n] = \det U[1, 2, \ldots, q|p + 1, \ldots, n]
= \det B[1, 2, \ldots, q|p + 1, \ldots, n] = d_p
$$
and
\[ C_p = L[q + 1, \ldots, p]V[q + 1, \ldots, p] \]
where
\[ v_{ij} = \det U[1, 2, \ldots, q, i | j, p + 1, \ldots, n], \quad i, j = 1 + 1, \ldots, p \]
while the corresponding matrix obtained from \( B \) is
\[ D_p = V[q + 1, \ldots, p]L[q + 1, \ldots, p]. \]
But \( C_p \) and \( D_p \) have the same eigenvalues, so that each upper (lower) bound appearing in (5) for \( A \) appears also in the corresponding bound for \( B \), and vice versa. Hence \( B \) has TP interval \( \mathcal{I}_A \).

**Corollary 2.2.** Suppose \( A \in M_n \) is TP, \( v \in \mathcal{I}_A \), \( A - vI = LU, \ B - vI = UL \), then \( B \) is TP with TP interval \( \mathcal{I}_A \).

**Proof.** If \( A \) has TP interval \( \mathcal{I}_A = (-a, b) \), then \( A - vI \) has TP interval \( (-a - v, b - v) \); \( B - vI \) has TP interval \( (-a - v, b - v) \); \( B \) has TP interval \( (-a, b) \).

It appears that it is not possible to extend Theorem 1 to matrices that are merely TN or NTN, as the following examples show. The matrix
\[
A = \begin{bmatrix}
0 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]
is TN, but \( A - \mu I \) is not TN for any \( \mu \neq 0 \).

Now we seek \( A \in M_n \) that is NTN but which has no interval around zero in which it is NTN. Take \( n = 5 \), so that
\[
A - \mu I = \begin{bmatrix}
a_{11} - \mu & a_{12} & a_{13} & a_{14} & a_{15} \\
a_{21} & a_{22} - \mu & a_{23} & a_{24} & a_{25} \\
a_{31} & a_{32} & a_{33} - \mu & a_{34} & a_{35} \\
a_{41} & a_{42} & a_{43} & a_{44} - \mu & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} - \mu
\end{bmatrix}
\]
Write \( G = A - \mu I \). Consider the two corner minors \( \det G[1, 2, 3|3, 4, 5] \) and \( \det G[2, 3, 4, 5|1, 2, 3, 4] \). We neek to make the former negative for all positive \( \mu \), and the latter negative for all negative \( \mu \). To do this, we need to make
\[
\det A[1, 2, 3|3, 4, 5] = 0, \quad \det A[1, 2|4, 5] > 0
\]
\[
\det A[2, 3, 4, 5|1, 2, 3, 4] = 0, \quad a_{51} > 0.
\]
Factorize \( A = LU \); these conditions will be satisfied if we can find \( L, U \), both NTN, such that
\[
\det U[1, 2, 3|3, 4, 5] = 0, \quad \det U[1, 2|4, 5] > 0
\]
\[
\det L[2, 3, 4, 5|1, 2, 3, 4] = 0, \quad l_{51} > 0.
\]
These conditions are satisfied by
\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}, \quad U = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 4 & 8 & 12 \\
0 & 0 & 1 & 3 & 5 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 5 & 9 & 13 \\
1 & 2 & 6 & 12 & 18 \\
1 & 2 & 6 & 13 & 21 \\
1 & 2 & 6 & 13 & 22
\end{bmatrix}.
\]

Now we have
\[
\det G[1, 2, 3|3, 4, 5] = -4\mu, \quad \det G[2, 3, 4|1, 2, 3, 4] = \mu^3.
\]

Note that \(A\) is not just NTN, it is O. This counterexample shows that even an oscillatory matrix need not have an interval \(I_A\) in which it is TN.

References